Guarded vs. Unguarded Iteration

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Introduction

- Guarded recursion:
  - Restrict recursive calls to appear under guarding operations (e.g. actions)
  - Obtain unique solutions
- Unguarded recursion:
  - Demand solutions to arbitrary recursive equations
  - Give up uniqueness
  - Instead impose equational laws
    - which are automatic under uniqueness
- Here: Unify guarded and unguarded iteration of side-effecting programs/processes
  - Side-effect = monad
  - Monads with guarded iteration = iterative monads
  - Monads with unguarded iteration = Elgot monads
  - Show (more or less) that

every Elgot monad is a quotient of an iterative monad

(FOSSACS 2017)

Guarded Recursion

Prevalent in process algebra; e.g.

$$P = a.P + b.P$$

has a unique solution because both recursive calls are guarded, i.e. appear under action prefixing.

What about

$$P = a.0 + P \qquad ?$$

- Semantics generates no transitions from $+P$
- Hence $P = a.0$

Try

$$P = Q +_{1/2} a.P \qquad Q = P +_{1/2} b.Q$$

Monads

- Monads formalize side-effecting functions $f : X \to TY$, e.g.
  - nondeterministic ($TX = \mathcal{P}X$)
  - partial ($TX = X + 1$)
  - state-based ($TX = S \to S \times X$)
- T is a type constructor for computations, with operations
  - $\eta : X \to TX$ (unit): Return a value
  - $(f : X \to TY) \mapsto (f^\star : TX \to TY)$ (lifting): Chain computations
- Kleisli category of $T : \mathbf{C} \to \mathbf{C}$ has

$$\text{morphisms } X \to Y = \mathbf{C}\text{-morphisms } X \to TY$$

– Laws for $\star$ guarantee identity / associativity laws

Completely Iterative Monads

model guarded recursion:

- Module for monad T:
  - Type constructor M (think 'terms with a guard on top')
  - lifting $(-)^\circ : \mathrm{Hom}(X, TY) \to \mathrm{Hom}(MX, MY)$
- Idealized monad = module-to-monad morphism $M \to T$
- $f : X \to T(Y + X)$ guarded $\iff$ factors through $Y + M(Y + Y)$
- T completely iterative $\iff$ every guarded $f : X \to T(X + Y)$ has a unique solution $f^\dagger : X \to TY$:

$$f^\dagger = [\eta, f^\dagger]^\star f.$$

- Examples: Infinite term monads $\nu\gamma.((-) + \Sigma\gamma)$.

Unguarded Recursion: Elgot Monads

(Complete) Elgot monad T:

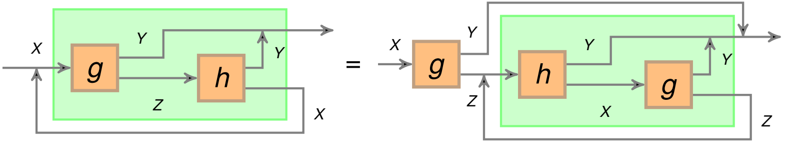
- Distinguishes solution $f^\dagger$ for every $f : X \to T(Y + X)$
- Solutions are in general non-unique
- Quasi-equational laws (dual to Bloom/Esik)

Examples:

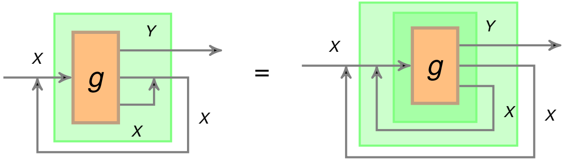
- Least fixpoints in cpo-enriched Kleisli-categories, e.g. $T = \mathcal{P}, (\text{–}) + 1, S \to \mathcal{P}(S \times (\text{–})), \dots$
- Extensions with free operations

Axioms for Iteration

Dinaturality:

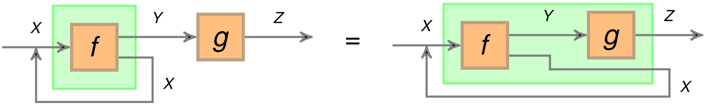


Codiagonal:

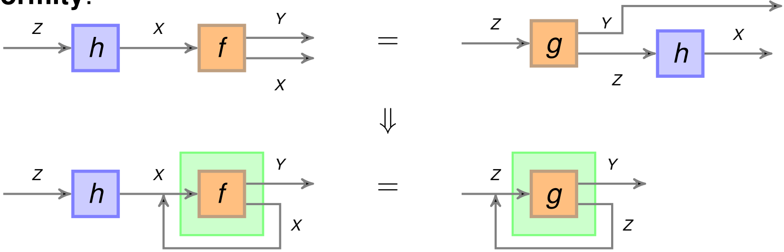


Axioms for Iteration, cont'd

Naturality:



Uniformity:



Adding Free Operations

Given

- monad T of effects
- functor Σ defining free operations

form

$$T_\Sigma^\nu = \nu\gamma.\, T((-) + \Sigma\gamma)$$

$\to$ side-effecting processes; e.g.

$$(\mathcal{P}_{\omega_1})_A^\nu = \nu\gamma.\, \mathcal{P}_{\omega_1}((-) + A \times \gamma)$$

is the denotational domain for countably branching processes.

- T_Σ^ν is a monad (Uustalu 2003)
- T_Σ^ν is completely iterative (Piróg/Gibbons MFPS 2014)
- T_Σ^ν inherits Elgotness from T (Goncharov/Rauch/LS MFPS 2015)

Abstract Guardedness

Abstractly guarded monad: Given coproduct injection $\sigma : Z \hookrightarrow Y$, distinguish (abstractly) σ-guarded Kleisli morphisms $X \to_\sigma TY$, satisfying

$$\textbf{(trv)}\quad \frac{f : X \to TY}{(T\,\mathsf{in}_1) \circ f : X \to_2 T(Y+Z)} \qquad \textbf{(wkn)}\quad \frac{f : X \to_\sigma TY}{f : X \to_{\sigma\theta} TY}$$

$$\textbf{(cmp)}\quad \frac{f : X \to_2 T(Y+Z) \qquad g : Y \to_\sigma TV \qquad h : Z \to TV}{[g,h] \diamond f : X \to_\sigma TV}$$

$$\textbf{(sum)}\quad \frac{f : X \to_\sigma TZ \qquad g : Y \to_\sigma TZ}{[f,g] : X+Y \to_\sigma TZ}$$

Abstract Guardedness: Examples

- Trivial guardedness: only immediately terminating definitions are guarded
- Total guardedness: everything is guarded
- Guardedness in idealized monads, when generalized to

  $f : X \to T(Y + X)$ inr-guarded $\iff$ f factors through $T(Y + M(Y + X))$

Guarded (Pre-)Iterative Monads

T abstractly guarded:

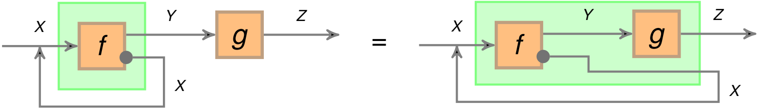
- T preiterative $\iff$
  T has solution $f^{\dagger}$ for every inr-guarded $f : X \to T(Y + X)$
- T iterative $\iff$ guarded morphisms have *unique* solutions

Laws have abstractly guarded versions;

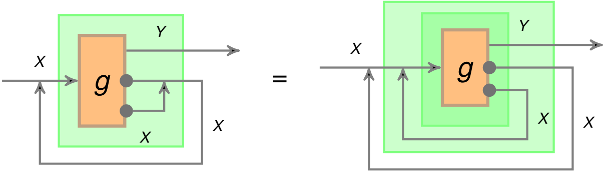
- laws are automatic for guarded iterative monads
- T Elgot $\iff$ T totally guarded preiterative & satisfies all laws.

Axioms for Abstractly Guarded Iteration

Naturality:

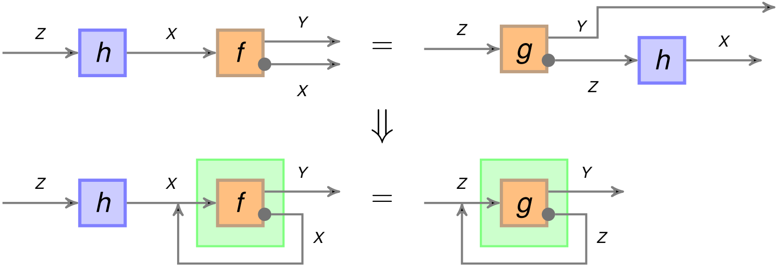


Codiagonal:



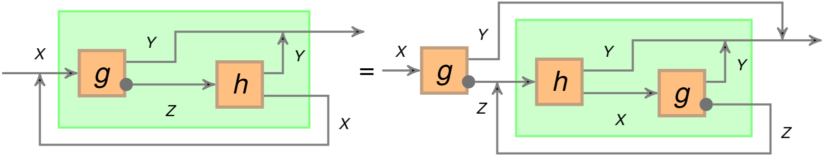
Axioms for Guarded Iteration, cont'd

Uniformity:

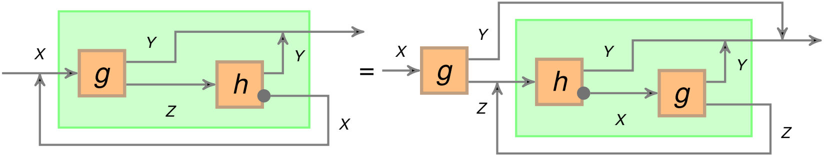


Axioms for Guarded Iteration, cont'd

Dinaturality (Variant 1):



Dinaturality (Variant 2):



Transferring Iteration

Iteration-congruent retraction $\rho : T \leftrightarrows S : \upsilon$:

- guarded monad morphism $\rho : T \to S$
- $\rho f = \rho g$ implies $\rho f^\dagger = \rho g^\dagger$.
- morphisms $\upsilon_X : SX \to TX$ (not necessarily natural) such that
  1. $\rho_X \upsilon_X = \mathrm{id}$
  2. $f : X \to_\sigma SY$ implies $\upsilon_Y f : X \to_\sigma TY$.

Transfer Theorem For T guarded pre-iterative and iteration-congruent retraction $\rho : T \leftrightarrows S : \upsilon$,

$$f^\ddagger := \rho\,(\nu f)^\dagger$$

defines an iteration operator on S that inherits all laws from T.

Main Result: Unguarded from Guarded Iteration

Theorem
Elgot monads =
totally guarded iteration-congruent retracts of guarded iterative monads.

Proof: '$\supseteq$': Immediate from transfer theorem.

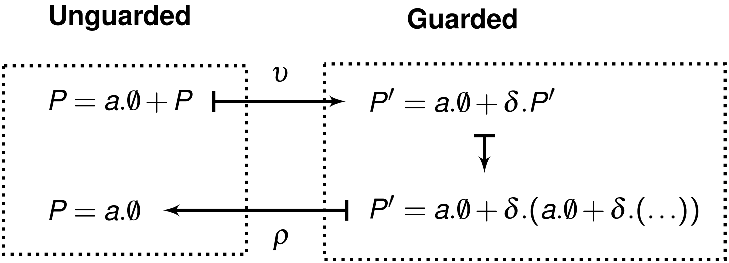
'$\subseteq$': Every Elgot monad S is an iteration-congruent retract of its coalgebraic transform

$$S^{\nu} = \nu\gamma.\, S(-+\gamma).$$

Example

For the process algebra monad $S = \nu\gamma.\, \mathcal{P}_{\omega 1}((-) + A \times \gamma)$:

$$S^{\nu} = \nu\gamma.\, S((-) + \{\delta\} \times \gamma) \cong \nu\gamma.\, \mathcal{P}_{\omega 1}((-) + (A + \{\delta\}) \times \gamma).$$



Conclusions

- Abstract notion of guardedness
  - subsumes standard guardedness as well as unguardedness
- Elgot monads = models of side-effecting unguarded iteration
- Have shown that

  every Elgot monad is an iteration-congruent retract of a guarded iterative monad,

  i.e.

  unguarded iteration arises by quotienting guarded iteration.

- Further results and applications:
  - Dinaturality follows from the other axioms
  - Simplified proof of Elgotness of T^{ν}_{Σ}
  - Sandwich theorem:
    Elgot monads are stable under sandwiching between adjoint functors
  - Elgot monads are the $(-)^{\nu}$-algebras that cancel delays

Further Work

- Quotienting Capretta's monad $\nu\gamma.\, X + \gamma$ (partiality/delay)
- Monads for infinite traces