One Eilenberg Theorem to Rule Them All

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joint work with Jiří Adámek, Liang-Ting Chen, Henning Urbat

January 11, 2017

Overview

Algebraic language theory:

Automata/languages **vs.** algebraic structures

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Categorical perspective:

$\mathrm{Id} \xrightarrow{\eta} T \xleftarrow{\mu} T^2$

- Automata via algebras and coalgebras.
- Languages via initial algebras and final coalgebras.
- Algebra via Lawvere theories and monads.

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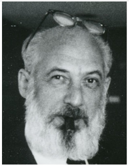
Our goal: Categorical Algebraic Language Theory!

Eilenberg's Variety Theorem (1976)



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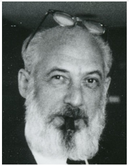

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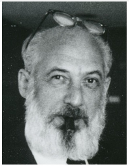

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Other types of languages:

- Weighted languages
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- Infinite words
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- Ordered words
  Bedon et. al. 1998, 2005
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This talk

A General Variety Theorem that covers them all!

Big Picture

General Variety Theorem

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Monads + **Duality**

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Gehrke, Grigorieff, Pin, ICALP 2008

Adámek, Milius, Myers, Urbat, FoSSaCS 2014, LICS 2015

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$$\mathbf{T}(\Sigma, \emptyset) = (\Sigma^+, \Sigma^\omega) \text{ on } \mathbf{Set}^2 \quad \text{and} \quad O = (\{0,1\}, \{0,1\}).$$

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- Weighted languages ($\mathcal{D}$ = vector spaces), tree languages ($\mathcal{D} = \mathbf{Set}^3$), cost functions ($\mathcal{D}$ = posets), ...

Algebraic recognition

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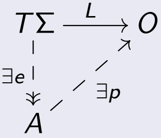

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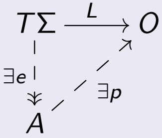

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Recognizable languages = regular ∞-languages

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Use **monads** to model the type of languages and the algebras recognizing them.

Use **duality** to relate varieties of languages to pseudovarieties of finite algebras.

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- Consider Stone duality between boolean algebras and Stone spaces:

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- This generalizes from $\mathbf{T}\Sigma = \Sigma^*$ to arbitrary monads $\mathbf{T}$!

General Duality

Monad $\mathbf{T}$ on $\mathcal{D}$ as before. Additionally, let $\mathcal{C}$ be a locally finite variety with:

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- Thus $\mathbf{Rec}(\Sigma)$ can be viewed as an object of $\mathcal{C}$!

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- For any *surjective map* $e : \Sigma^* \twoheadrightarrow A$,

e carries a *quotient monoid* of Σ^* $\iff$ all $\Sigma^* \xrightarrow{x(-)y} \Sigma^*$ lift along e.

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{x(-)y} & \Sigma^* \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} \\ A & \xrightarrow[\exists]{} & A \end{array}$$

Derivatives: General Case

Definition

Unary presentation $\mathbb{U} = \{\, T\Sigma \xrightarrow{u} T\Sigma \,\}$: for any quotient $e : T\Sigma \twoheadrightarrow A$,

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Definition

For a language $T\Sigma \xrightarrow{L} O$ and $T\Sigma \xrightarrow{u} T\Sigma$ in $\mathbb{U}$, we have the **derivative**

$$u^{-1}L := (\; T\Sigma \xrightarrow{u} T\Sigma \xrightarrow{L} O \;).$$

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$$\left(\begin{array}{c} \textbf{varieties of} \\ \textbf{languages} \end{array} \right) \cong \left(\begin{array}{c} \textbf{pseudovarieties of} \\ \textbf{T-algebras} \end{array} \right)$$

Variety of languages

For each alphabet Σ a subobject $V_\Sigma \subseteq \mathbf{Rec}(\Sigma)$ in $\mathcal{C}$ closed under derivatives and $\mathbf{T}$-preimages.

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How to prove the theorem?

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How to prove the theorem?

Dualize!

Applications

$\mathcal{C}^{op} \cong \hat{\mathcal{D}}$ $\qquad$ $\mathbf{T}$ $\qquad$ $\mathbb{U}$ $\qquad$ $\cdots$

$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$

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More than a dozen variety theorems known in the literature.

Some results covered by the General Variety Theorem

Languages of finite words:

- $\cup$, $\cap$, $(-)^{\complement}$
  Eilenberg 1976
- Only $\cup$, $\cap$
  Pin 1995
- Only $\cup$
  Polák 2001
- Only $\oplus$
  Reutenauer 1980
- Fewer monoid morphisms
  Straubing 2002
- Fixed alphabet, no preimages
  Gehrke, Grigorieff, Pin 2008

Other types of languages:

- Weighted languages
  Reutenauer 1980
- Infinite words
  Wilke 1991, Pin 1998
- Ordered words
  Bedon et. al. 1998, 2005
- Ranked trees
  Almeida 1990, Steinby 1992
- Binary trees
  Salehi, Steinby 2008
- Cost functions
  Daviaud, Kuperberg, Pin 2016

Applications

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More than a dozen variety theorems known in the literature.

New results, e.g. extending work of Gehrke, Grigorieff, Pin (2008) from finite words to infinite words, trees, cost functions, ...

Conclusions and Further Work

Eilenberg = Monads + Duality

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- Monadic second order logic for a monad ?