PROPS FOR CONCURRENCY AND QUANTUM INFORMATION

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Interacting Bialgebras are Frobenius, FoSSaCS `14

IFIP WG1.3 10/01/14

Hothorpe Hall, Theddingworth

PROPS, INTUITIVELY

S. Mac Lane. Categorical algebra. Bull Amer Math Soc, 71:40–106, 1965.

S. Lack. Composing PROPs. Theor App Categories, 13(9):147–163, 2004.

- Lawvere theories are a way to study algebraic theories categorically
  - the objects of Lawvere theory are “variables”
  - arrows $n \to k$ are k-tuples with n variable
- PROPs are a “linear version” where you cannot duplicate nor discard variables
- This allows the study of theories that feature **both** algebraic and coalgebraic operations

PRO

- PROs = strict monoidal categories with objects the natural numbers, tensor product on objects = addition
  - PRO **P** of permutations
    - no arrows $n \to k$ if n different from k
    - otherwise arrows are permutations $[n] \to [n]$, composition is as expected
  - morphisms of PROs = strict identity-on-objects monoidal functors

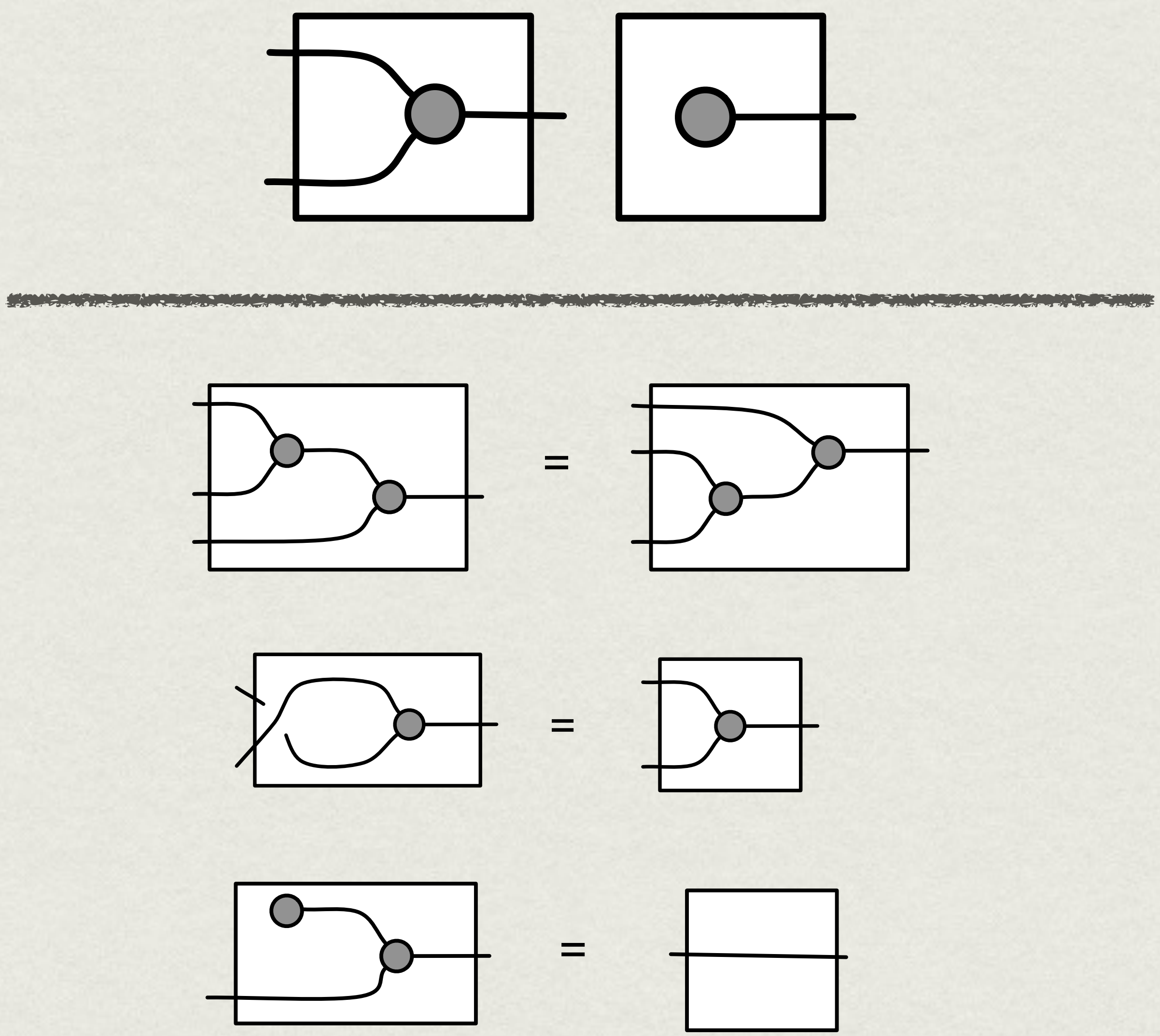
PROP

- the category of PROPs is **P/PRO**
- to give a PROP we need to
  - give a PRO
  - identify the permutations
  - make sure that they behave like (satisfy the same equations as) permutations on finite sets

THE PROP OF FUNCTIONS F

- arrows n → m are functions [n] → [m]
  - identities and composition are as expected
  - the PROP permutations are the permutations

PROP OF COMMUTATIVE MONOIDS

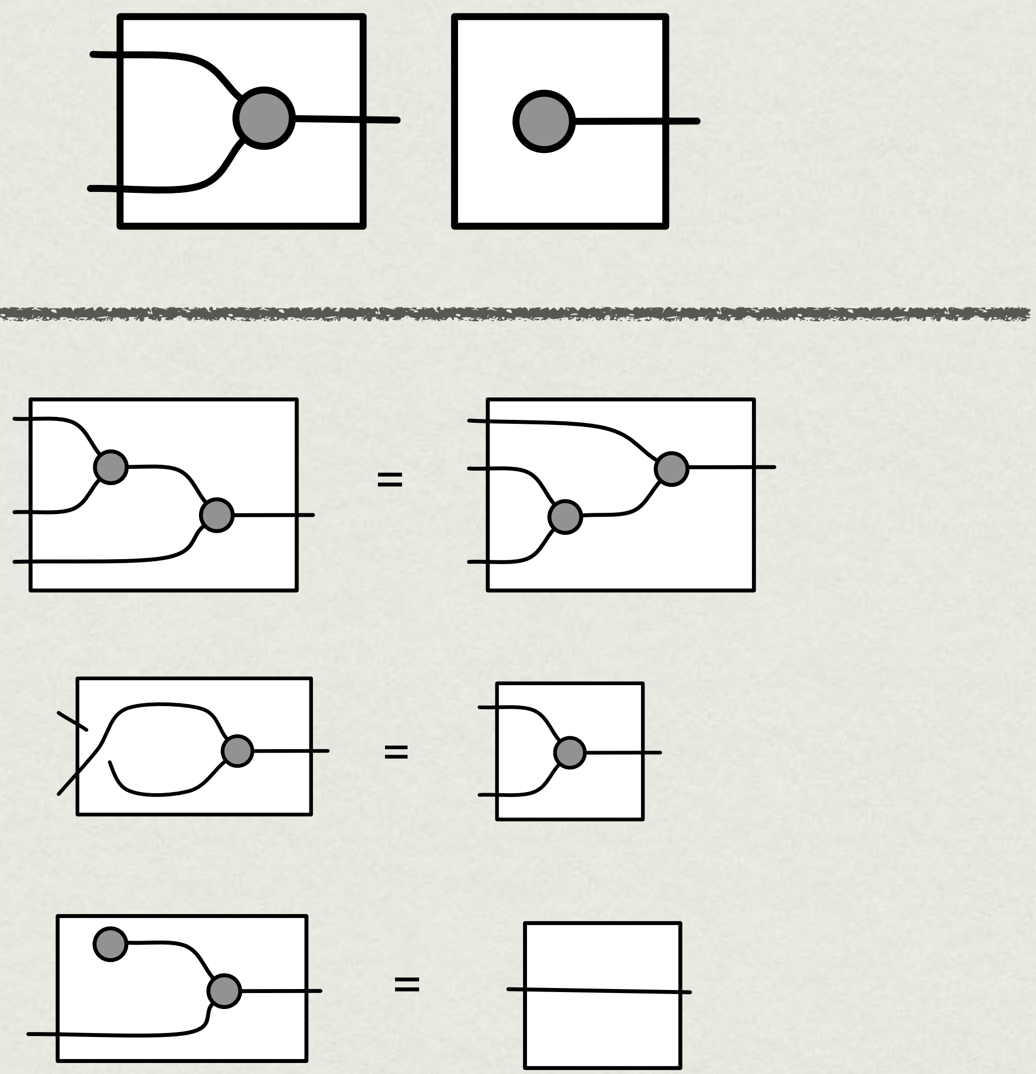


Observation:
The free PROP on these equations is isomorphic to the PROP **F** of functions

TWO KINDS OF PROPS

- “Semantic PROPs” - eg. the PROP **F** of functions
- “Syntactic PROPs” - eg. the PROP of commutative monoids
  - freely generated from a set of generators (the syntax) modulo a set of equations

PROP OF COCOMMUTATIVE COMONOIDS



Observation:
The free PROP is isomorphic to the PROP of $\mathbf{F}^{op}$

COMPOSING PROPS

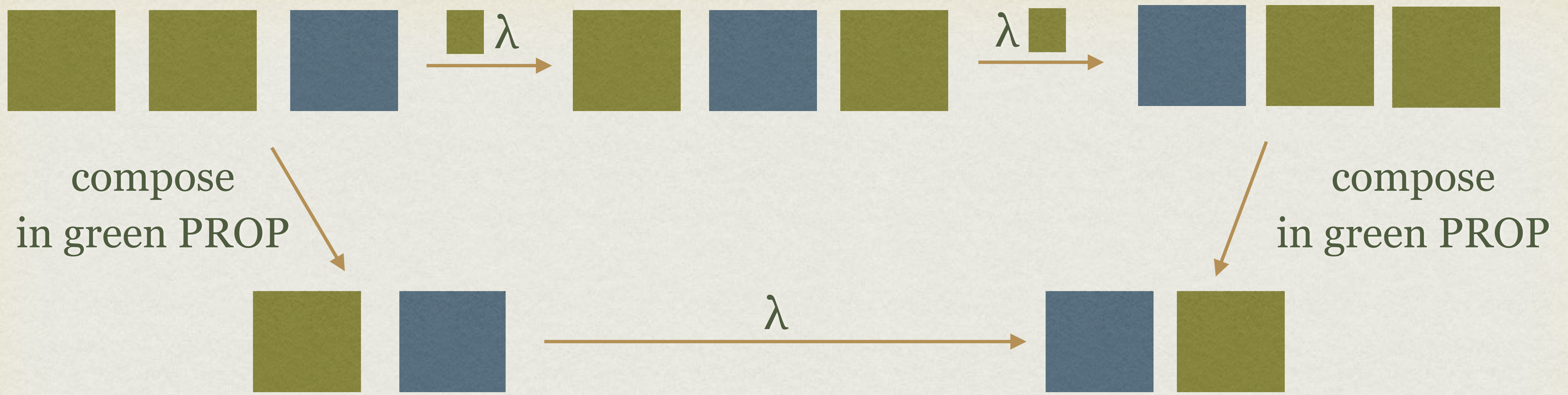
S. Lack. Composing PROPs. Theor App Categories, 13(9):147–163, 2004.

- Monads are not always functors, the theory of monads works in any 2-category, not just **Cat**

  R. Street. The formal theory of monads. J Pure Appl Algebra, 2(1):243–265, 2002.

- A monad is a 1-cell and two 2-cells, satisfying the triangle equations
- (small) category = monad in Span(**Set**)
  - multiplication = composition or arrows, identity = pick out identity arrows
- So categories (with the same object set) can be composed as 1-cells
  - the resulting span of sets can be given a categorical structure if there is a distributive law

DISTRIBUTIVE LAW - THE MEAT



EXAMPLE 1 - SPANS

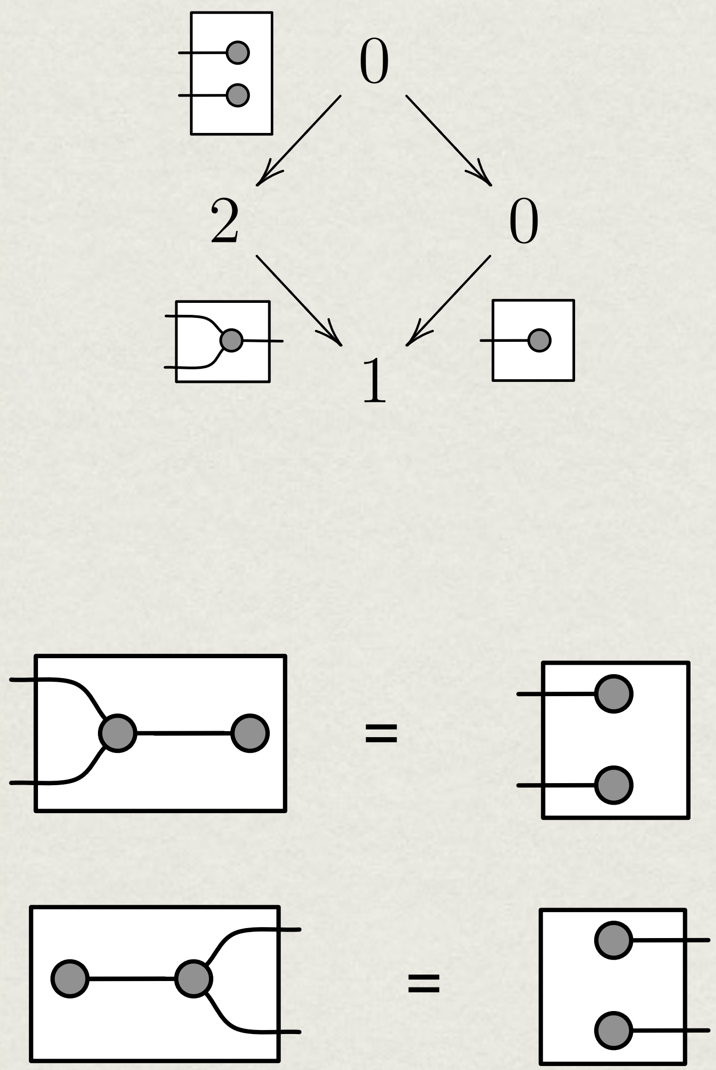
S. Lack. Composing PROPs. Theor App Categories, 13(9):147–163, 2004.

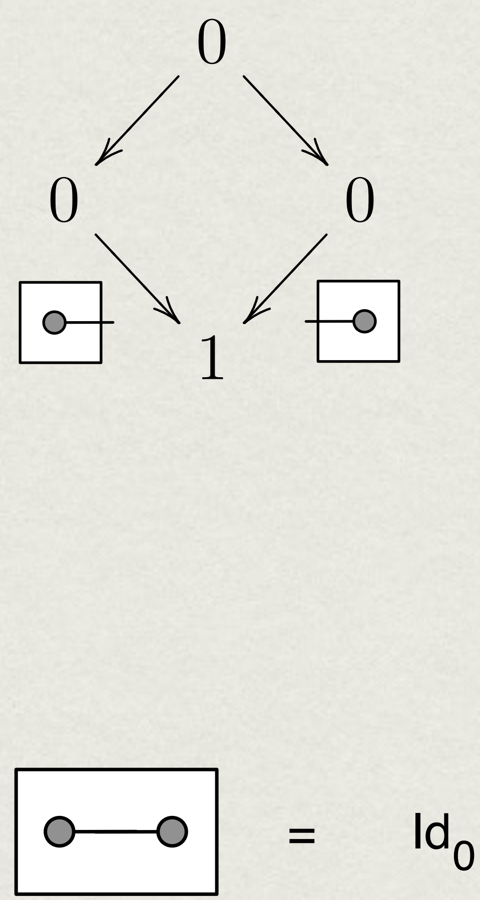
Pb: $\mathbf{F} \,;\, \mathbf{F}^{op} \to \mathbf{F}^{op} \,;\, \mathbf{F}$

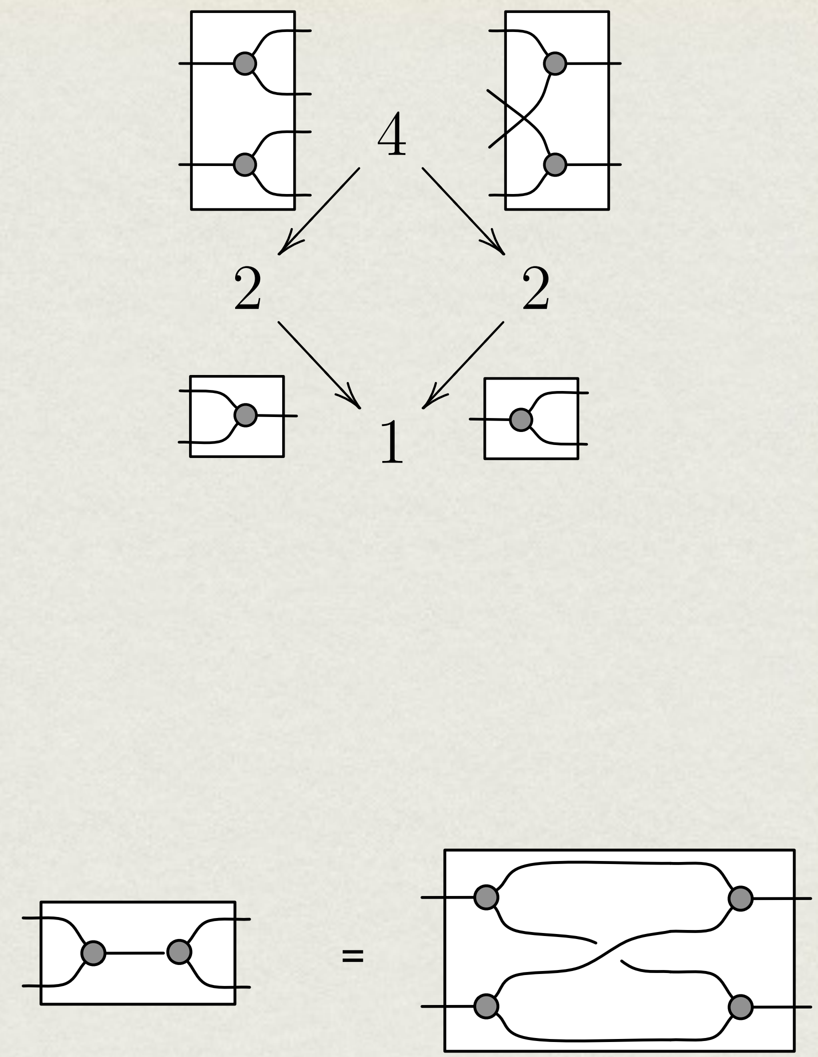
$\longrightarrow \; \longleftarrow \qquad \longleftarrow \; \longrightarrow$

- The universal properties of pullbacks guarantee that this indeed defines a distributive law
- Makes $\mathbf{F}^{op};\mathbf{F}$ into a PROP - the PROP of spans of finite sets (isomorphic spans are identified)

READING THE AXIOMS

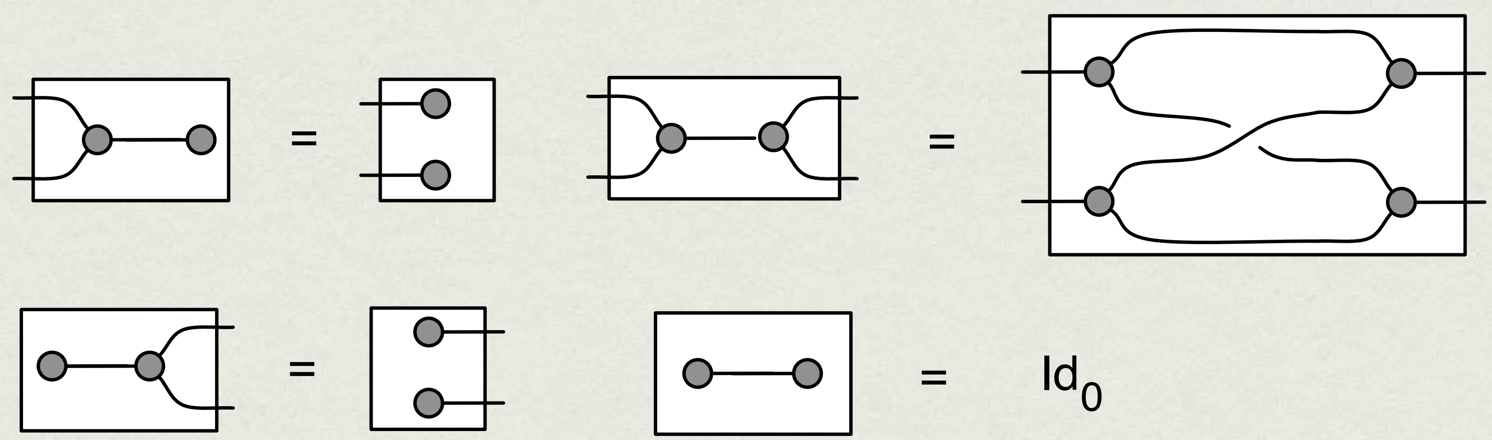






Any other pullback is a coproduct of these basic ones

A THEORY OF SPANS



- = the theory of commutative bialgebra
- **Corollary:** free PROP on equations above is isomorphic to the PROP of spans

EXAMPLE 2 - COSPANS

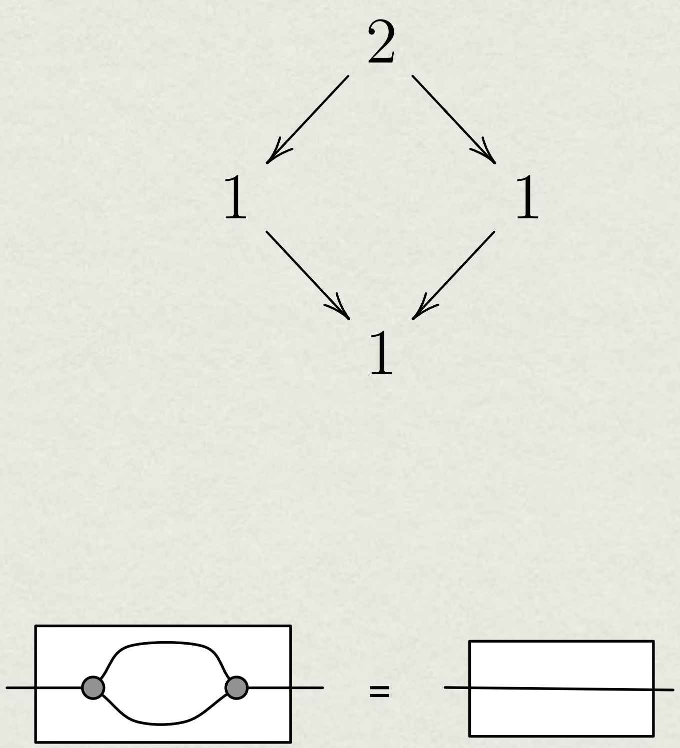
S. Lack. Composing PROPs. Theor App Categories, 13(9):147–163, 2004.

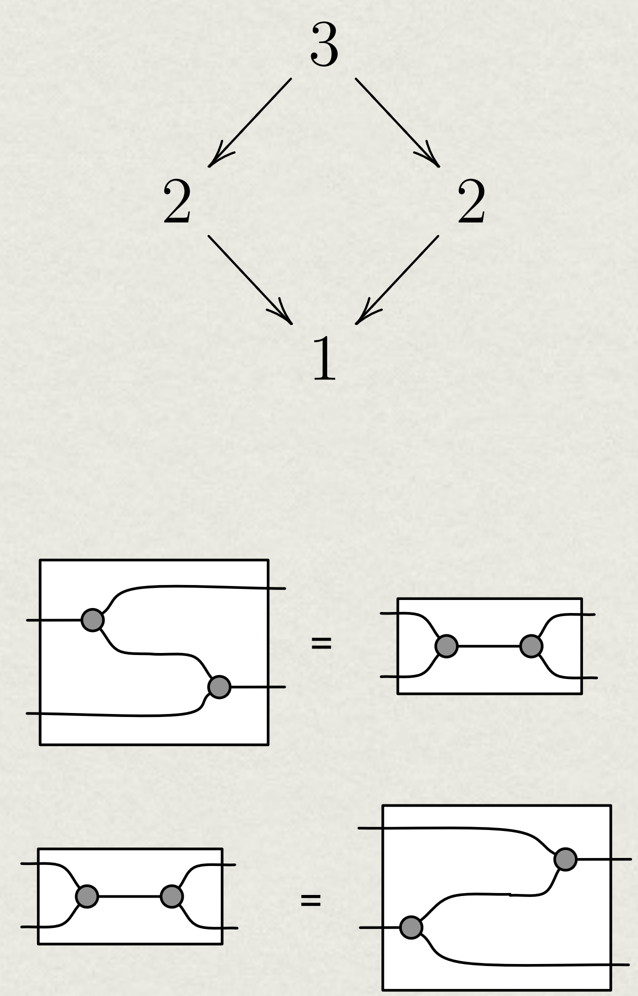
$$\text{Po}: \mathbf{F}^{op} ; \mathbf{F} \rightarrow \mathbf{F} ; \mathbf{F}^{op}$$

$$\leftarrow \; \rightarrow \qquad \rightarrow \; \leftarrow$$

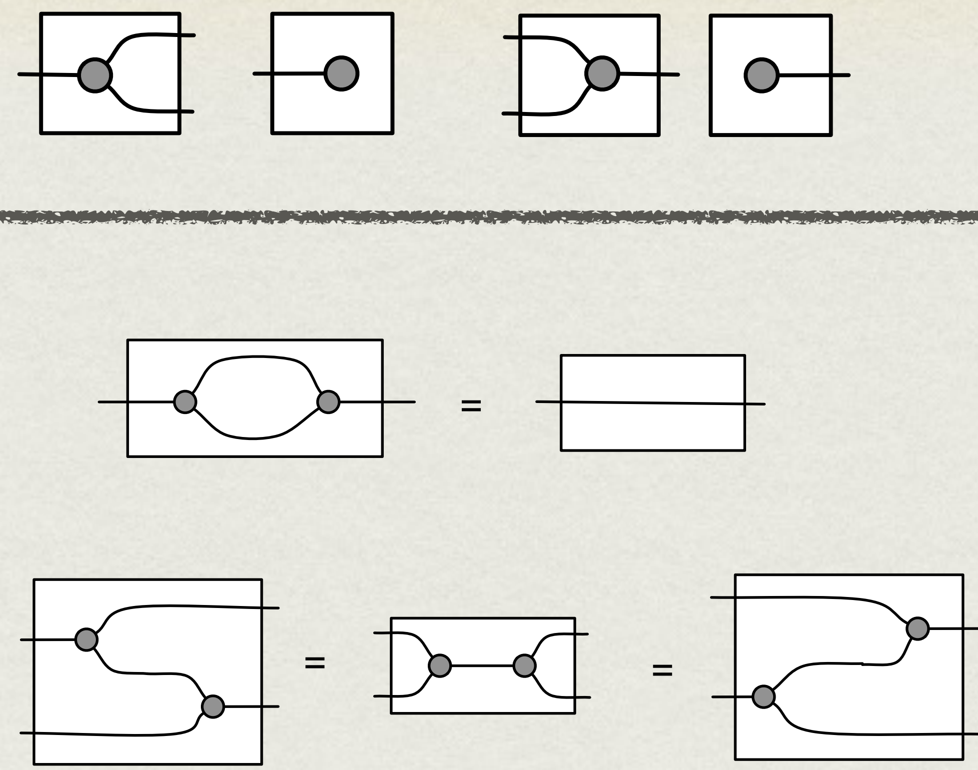
- The universal properties of pushouts guarantee that this indeed defines a distributive law
- Makes $\mathbf{F} ; \mathbf{F}^{op}$ into a PROP - the PROP of cospans of finite sets (isomorphic cospans are identified)

READING THE AXIOMS





A THEORY OF COSPANS



- = the theory of separable Frobenius algebra
- **Corollary:** the free PROP on the equations above is isomorphic to the PROP of cospans

THE PROP OF EQUIVALENCE RELATIONS

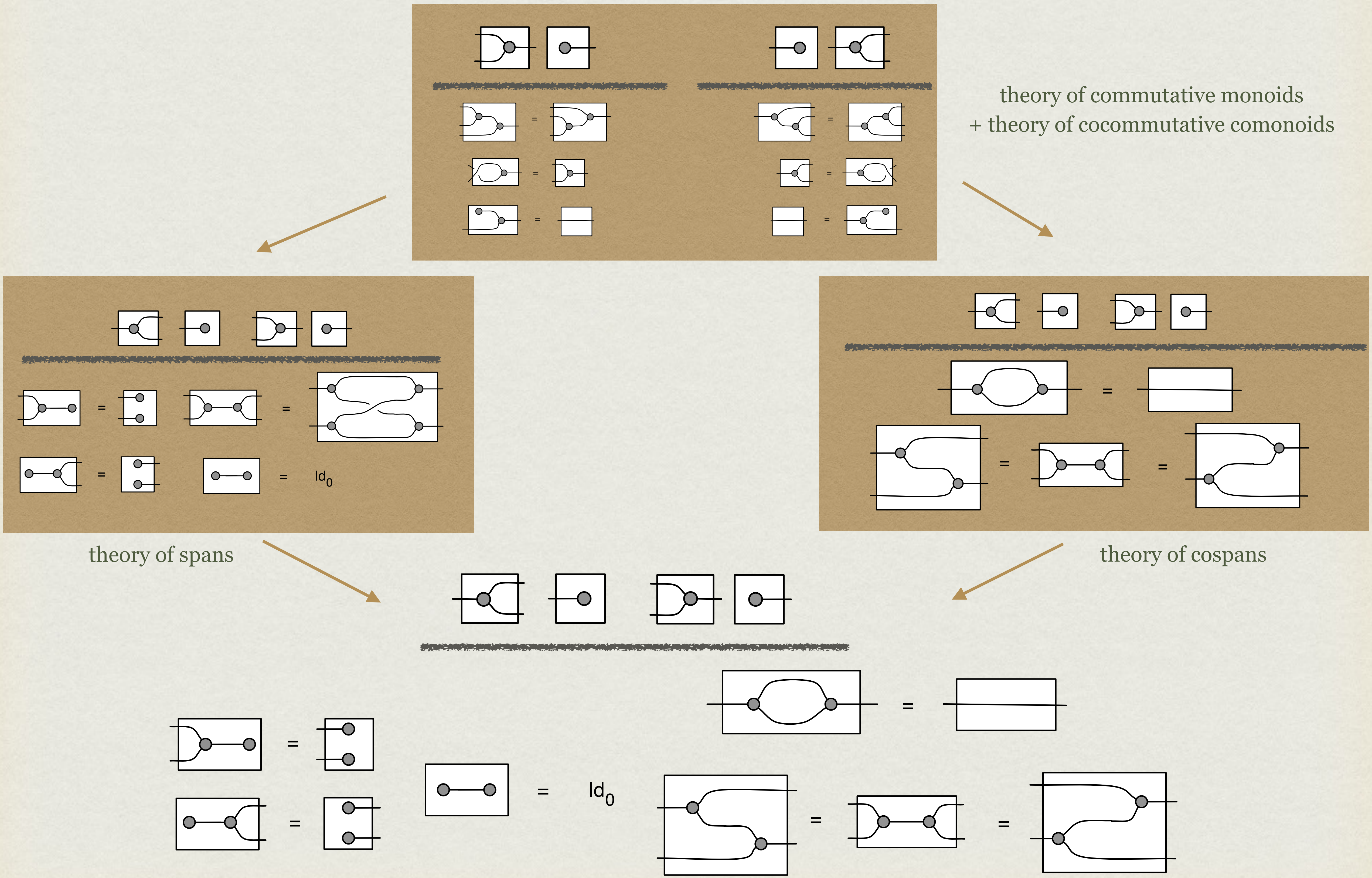
- arrows n $\to$ m are equivalence relations on n+m
  - composition is relational
  - the PROP permutations are the graphs of permutations

GLUING PROPS

$$
\begin{array}{ccc}
 & \mathbf{F} + \mathbf{F}^{op} & \\
\swarrow & & \searrow \\
\mathsf{Span}(\mathbf{F}) & & \mathsf{Cospan}(\mathbf{F}) \\
\searrow & & \swarrow \\
 & \mathbf{EqRel} &
\end{array}
$$

- The PROP of equivalence relation is given by the pushout (in the category of PROPs) above

- Thus the theory of equivalence relations is given by a pushout of the theories

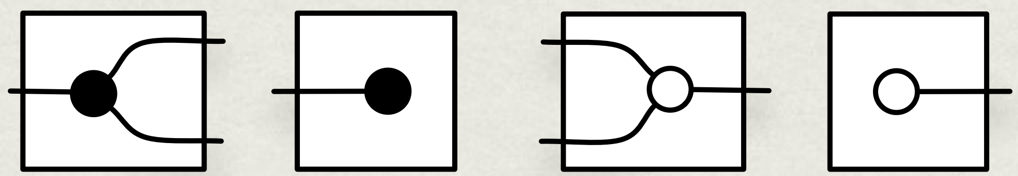


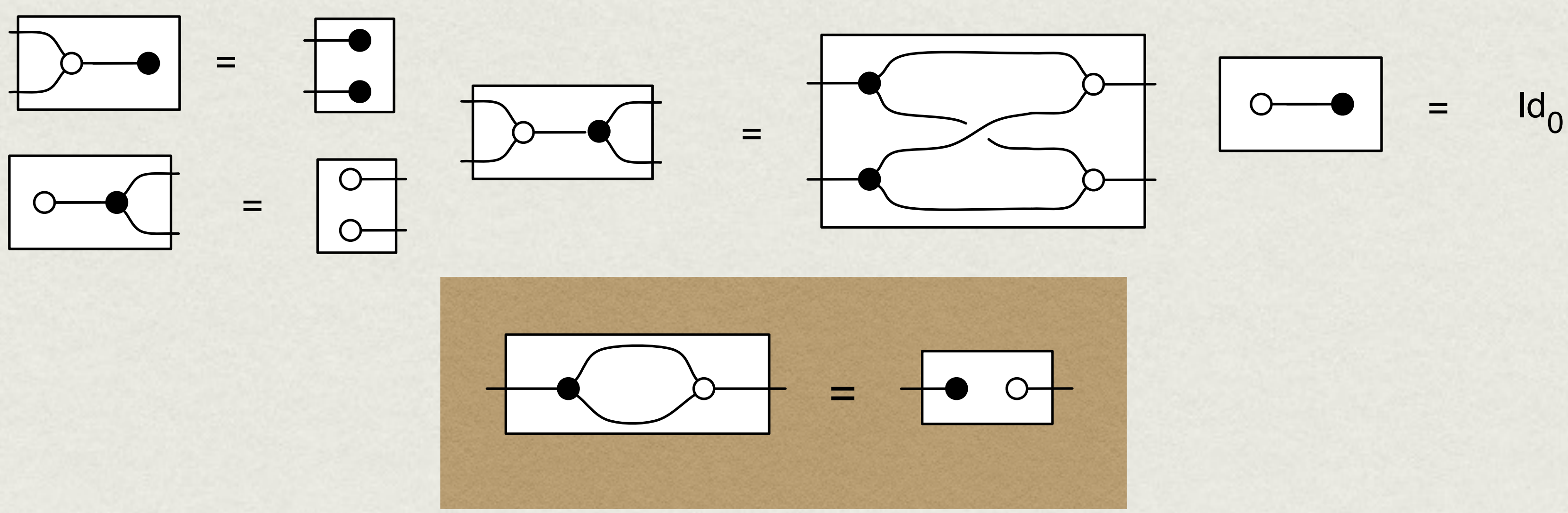
THE PROP OF Z_2 MATRICES

- Mat Z_2: arrows $n \to m$ are functions $m \times n$ matrices with entries from $Z_2 = \{0, 1\}$
  - composition is matrix multiplication
  - the PROP permutations are rearrangements of the rows of the identity matrix
- Equivalent to the category of f.d. Z_2 vector spaces

THEORY OF Z_2 MATRICES

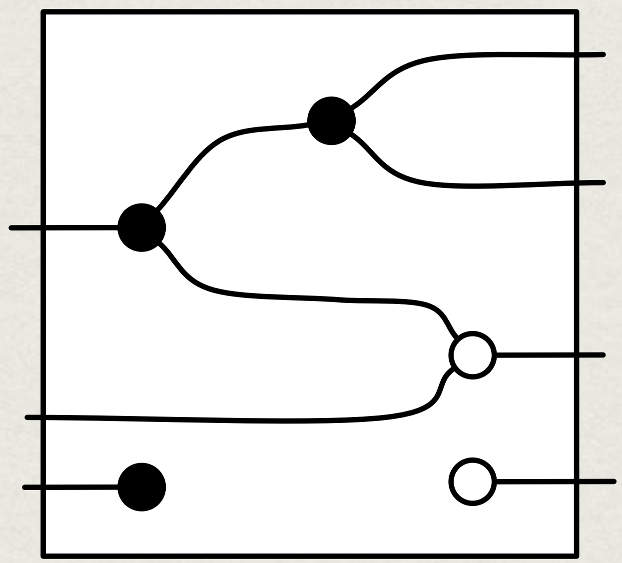
Y. Lafont. Towards an algebraic theory of boolean circuits. J Pure Appl Alg, 184:257–310, 2003.





- The free PROP **AB** is isomorphic to Mat Z_2

EXAMPLE



$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

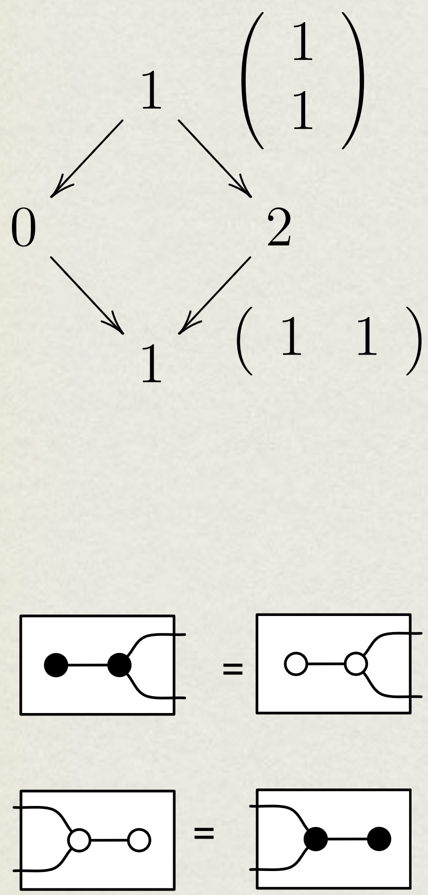
SPANS OF MATRICES

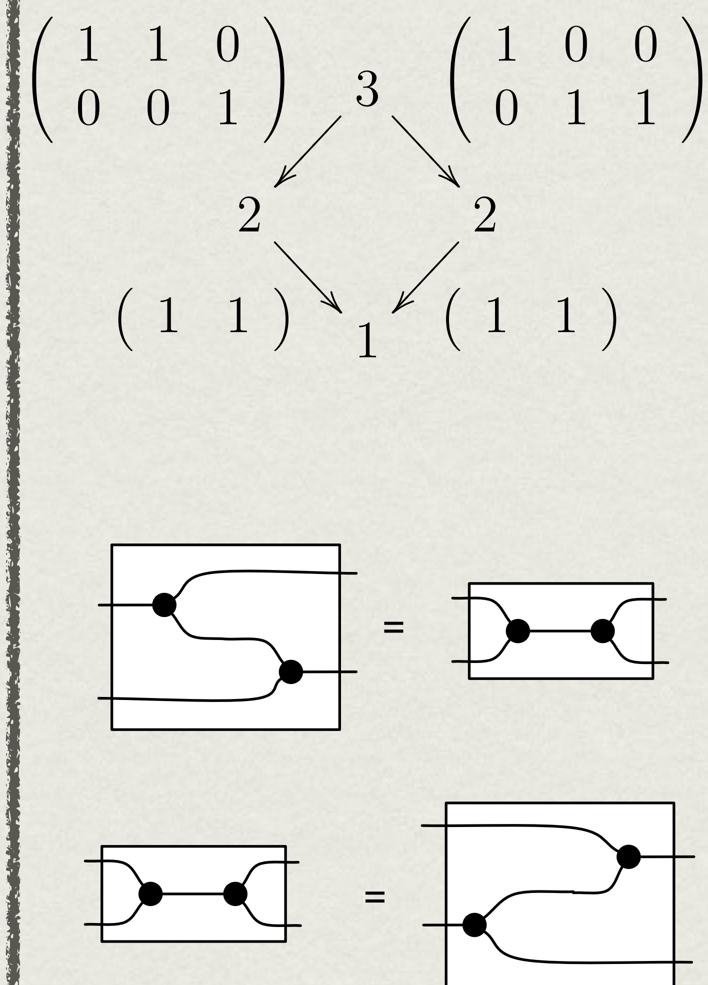
$$\mathrm{Pb}: \mathrm{Mat}\, Z_2 \,;\, (\mathrm{Mat}\, Z_2)^{op} \rightarrow (\mathrm{Mat}\, Z_2)^{op} \,;\, \mathrm{Mat}\, Z_2$$

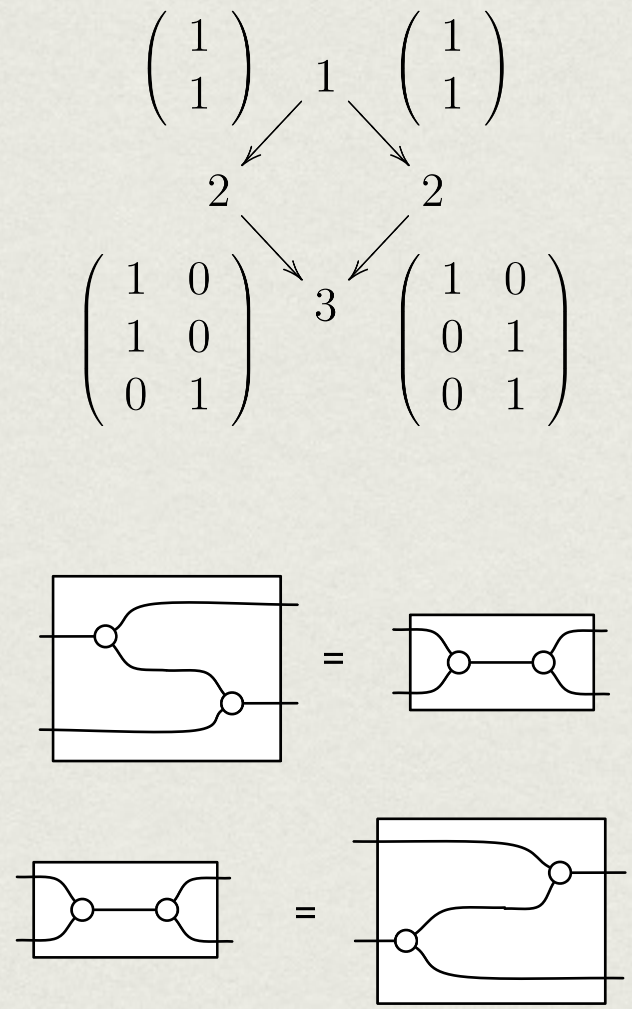
$$\rightarrow \leftarrow \qquad\qquad\qquad \leftarrow \rightarrow$$

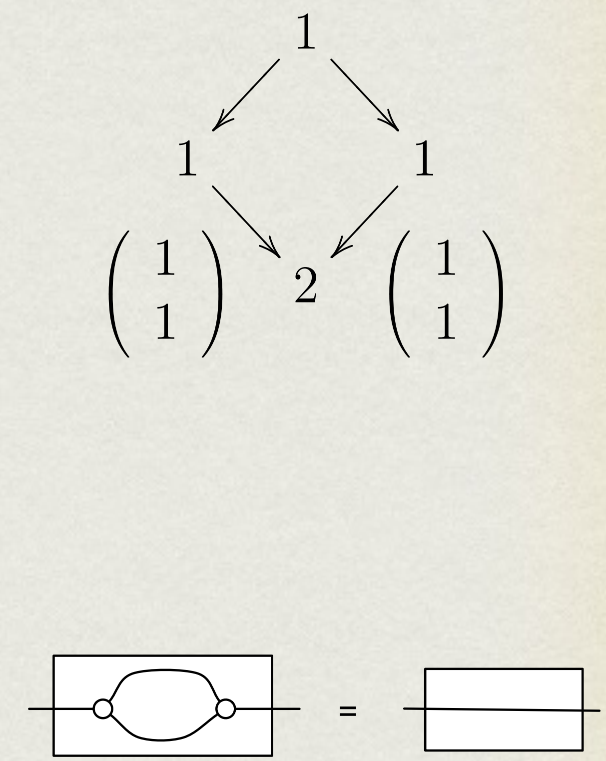
- The universal properties of pullbacks guarantee that this indeed defines a distributive law
- Makes $(\mathrm{Mat}\ Z_2)^{op}$; $\mathrm{Mat}\ Z_2$ into a PROP - the PROP of spans of Z_2 matrices

READING THE AXIOMS

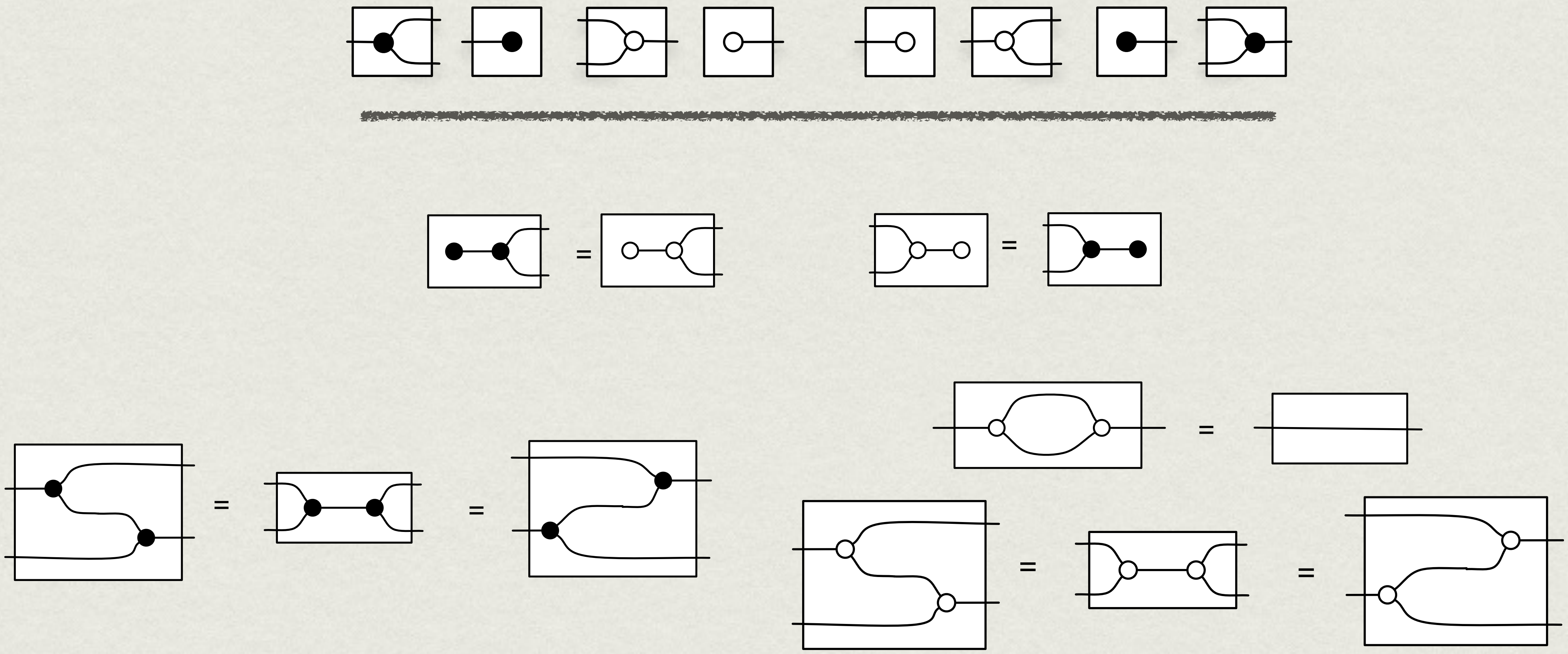








THEORY OF SPANS OF Z_2 MATRICES



- The free PROP is isomorphic to the PROP of spans of Z_2 matrices

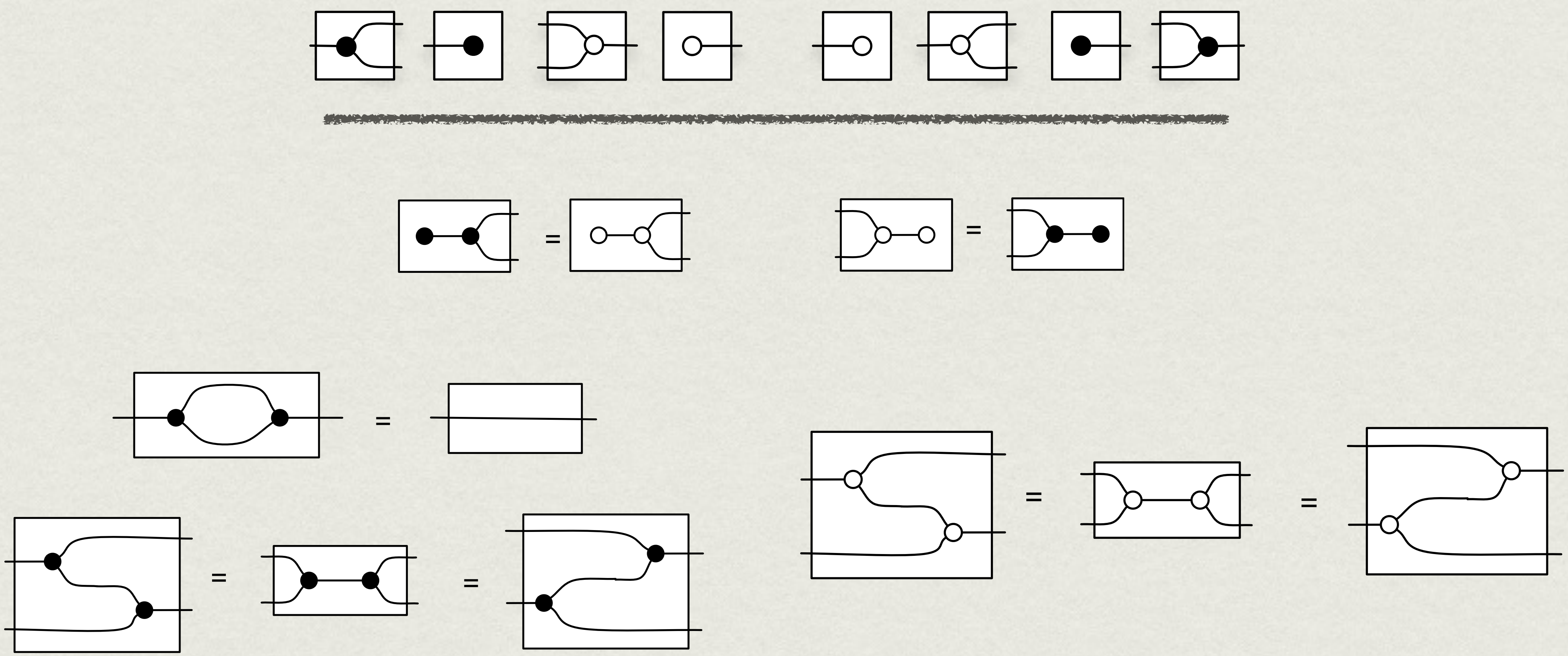
COSPANS OF MATRICES

$$\text{Po}: (\text{Mat } Z_2)^{op} \; ; \; \text{Mat } Z_2 \rightarrow \text{Mat } Z_2 \; ; \; (\text{Mat } Z_2)^{op}$$

$$\longleftarrow \; \longrightarrow \qquad\qquad \longleftarrow \; \longrightarrow$$

- The universal properties of pushouts guarantee that this indeed defines a distributive law
- Makes Mat Z_2 ; (Mat Z_2)^{op} into a PROP - the PROP of cospans of Z_2 matrices

THE THEORY OF COSPANS OF MATRICES

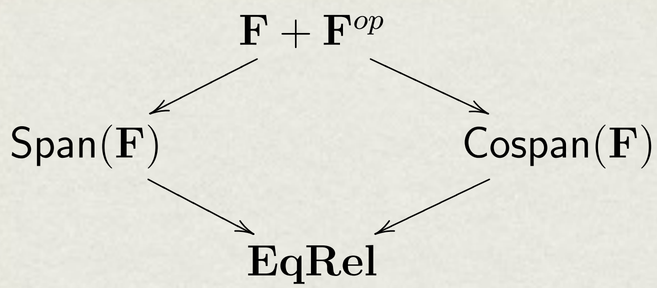


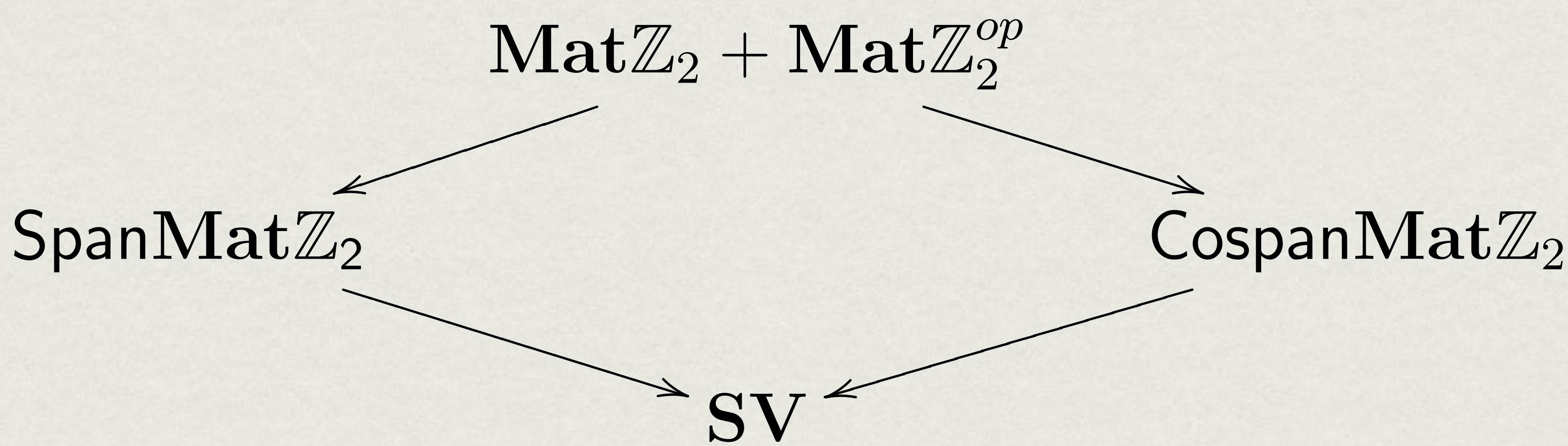
- The free PROP is isomorphic to the PROP of cospans of Z_2 matrices

Z_2 VECTOR SUBSPACES

- **SV**: arrows $n \to m$ are subspaces of $Z_2^n \times Z_2^m$
  - identities and composition are as expected
  - the PROP permutations are the subspaces "generated by permutations"

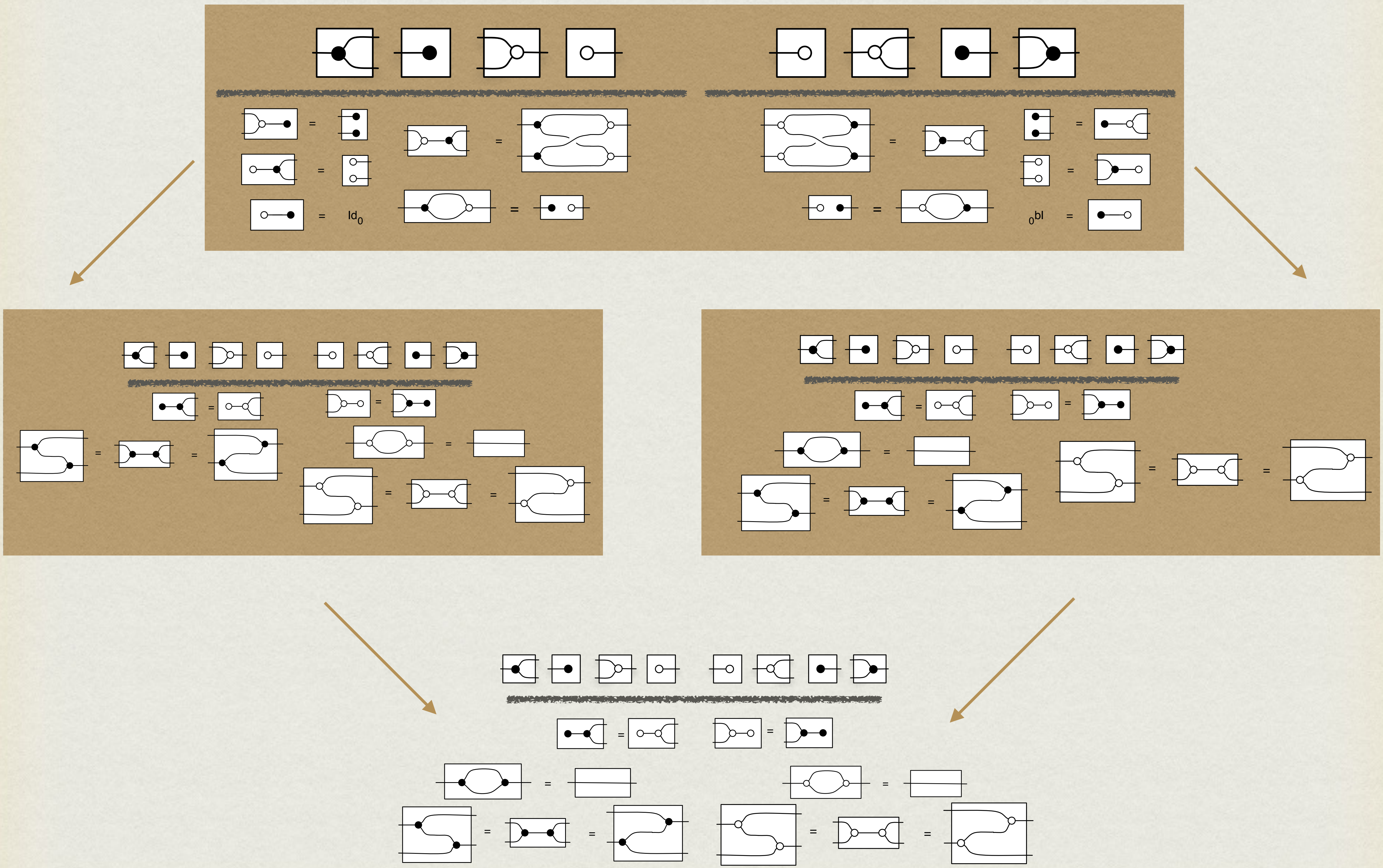
GLUING PROPS



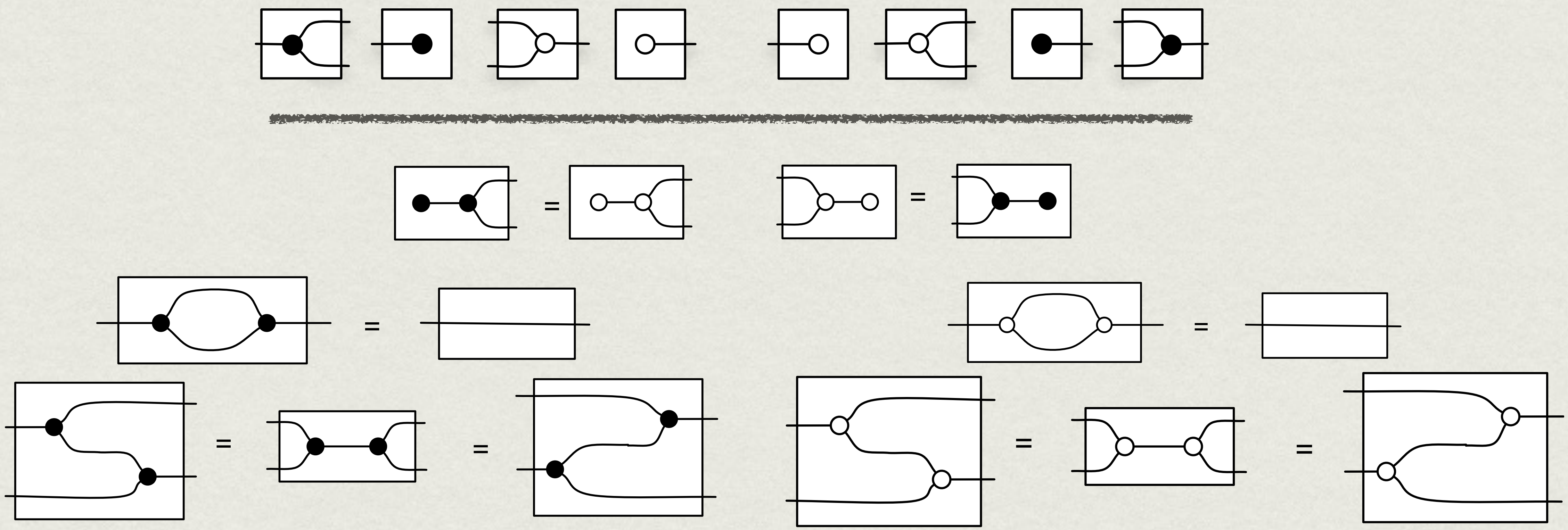


- so Z2 vector subspaces arise via a similar construction to equivalence relations!

So the theory of Z2 vector subspaces is given by a pushout of theories



THE THEORY OF INTERACTING BIALGEBRAS **IB**



- The free PROP is isomorphic to the PROP of spans of Z_2 vector subspaces

OTHER NAMES FOR **IB**

B. Coecke and R. Duncan. Interacting quantum observables. In ICALP'08, pages 298–310, 2008.

B. Coecke, R. Duncan, A. Kissinger, and Q. Wang. Strong complementarity and non-locality in categorical quantum mechanics. In LiCS'12, pages 245–254, 2012.

- A sub-calculus of the ZX-calculus (Coecke and Duncan) for quantum things

R. Bruni, I. Lanese, and U. Montanari. A basic algebra of stateless connectors. Theor Comput Sci, 366:98–120, 2006.

- A tweak of the calculus of stateless connectors

    - is this an amazing coincidence or is there something deeper here?

FUTURE WORK

- Slogan: compositional theories for concurrency and quantum information are often both algebraic and colgebraic
  - so we need something like PROPs to understand them
- Other examples (there are many)
  - graphical linear algebra!
- the theory of Petri nets with boundaries
  - axiomatising behavioural equivalences
- understand more deeply the connections with Quantum information