An Institution-Independent Approach to Logic Programming

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Goal

to describe an axiomatic approach to logic programming based on which the conventional variant of the paradigm [Lloyd, 1987] can be explored for other formalisms

- order-sorted [Goguen and Meseguer, 1986]
- higher-order [Meseguer, 1989]
- constraint [Diaconescu, 2000]
- behavioural [Goguen et al., 2002]
- service-oriented [Țuțu and Fiadeiro, 2013]

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founded on a three-level hierarchy of concepts meant to capture

- the denotational semantics of logic programming, based on a notion of generalised substitution system
- the operational semantics of logic programming, supported by a notion of logic-programming framework
- the various constructions of logic programs through a notion of logic-programming language

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suitable for investigating

- properties concerning the satisfaction of quantified sentences
- an abstract variant of Herbrand's theorem
- a resolution-based procedure for computing solutions to queries

Conventional Logic Programming

signature

  op $\mathtt{0}\colon 0$

  op $\mathtt{s\_}\colon 1$

  pred $\mathtt{add}\colon 3$

axioms

  $\mathtt{add}(\mathtt{0}, M, M) \xleftarrow[M]{}$

  $\mathtt{add}(\mathtt{s}\,M, N, \mathtt{s}\,P) \xleftarrow[M,N,P]{} \mathtt{add}(M, N, P)$

$\xmapsto[X_1]{} \mathtt{add}(\mathtt{s}\,\mathtt{0}, \mathtt{s}\,\mathtt{0}, X_1)$

module $\mathtt{NAT}$ = **free**

  sort $\mathtt{Nat}$

  op $\mathtt{0}\colon \ \to \mathtt{Nat}$

  op $\mathtt{s\_}\colon \mathtt{Nat} \to \mathtt{Nat}$

module $\mathtt{ADD}$ = $\mathtt{NAT}$ **then**

  op $\_ + \_\colon \mathtt{Nat}\,\mathtt{Nat} \to \mathtt{Nat}$

  cl $\mathtt{0} + M = M \xleftarrow[M\,:\,\mathtt{Nat}]{}$

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$\xmapsto[X_1\,:\,\mathtt{Nat}]{} \mathtt{s}\,\mathtt{0} + \mathtt{s}\,\mathtt{0} = X_1$

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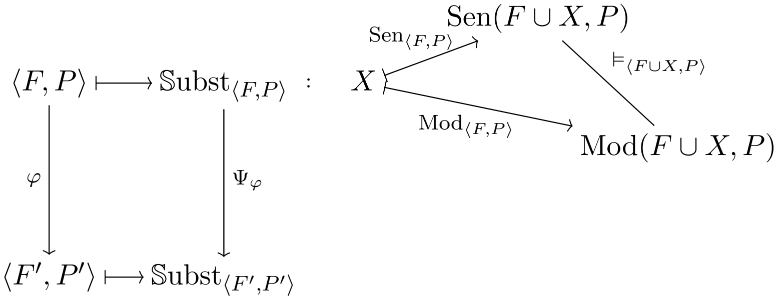
The Relational Setting

$$\langle F, P \rangle \longmapsto \mathbb{S}\mathrm{ubst}_{\langle F,P \rangle} \; : \quad X \xrightarrow{\mathrm{Sen}_{\langle F,P\rangle}} \mathrm{Sen}(F \cup X, P), \qquad X \xrightarrow{\mathrm{Mod}_{\langle F,P\rangle}} \mathrm{Mod}(F \cup X, P), \qquad \mathrm{Sen}(F \cup X, P) \;\overset{\models_{\langle F \cup X, P\rangle}}{—}\; \mathrm{Mod}(F \cup X, P)$$

The Relational Setting



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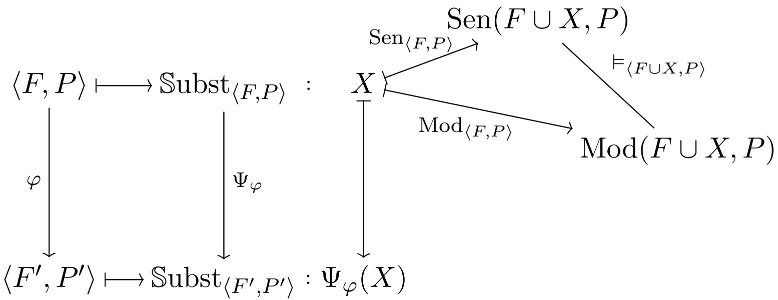

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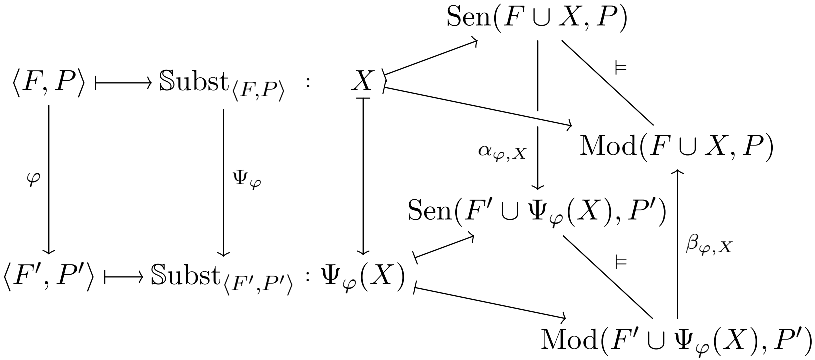

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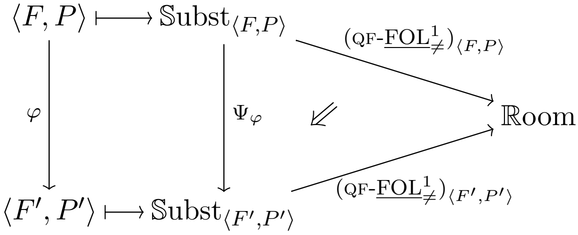

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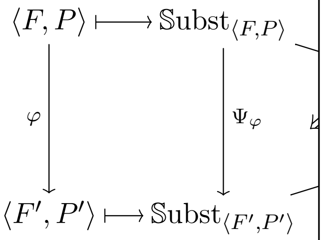
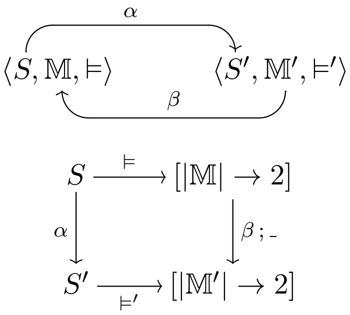


Rooms



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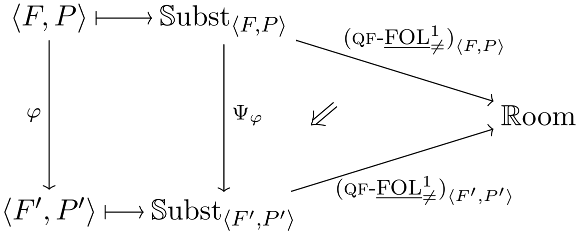

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$$\begin{array}{ccc} & \mathrm{Sen}(F,P) & \\ {\scriptstyle \alpha_{\langle F,P\rangle,X}}\swarrow & & \searrow{\scriptstyle \alpha_{\langle F,P\rangle,Y}} \\ \mathrm{Sen}_{\langle F,P\rangle}(X) & \xrightarrow[\psi(\_)]{} & \mathrm{Sen}_{\langle F,P\rangle}(Y) \end{array}$$

$$\begin{array}{ccc} & \mathrm{Mod}(F,P) & \\ {\scriptstyle \beta_{\langle F,P\rangle,X}}\nearrow & & \nwarrow{\scriptstyle \beta_{\langle F,P\rangle,Y}} \\ \mathrm{Mod}_{\langle F,P\rangle}(X) & \xleftarrow[\_\restriction_{\psi}]{} & \mathrm{Mod}_{\langle F,P\rangle}(Y) \end{array}$$

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Substitution systems

$$\langle F,P\rangle \longmapsto \mathbb{S}\mathrm{ubst}_{\langle F,P\rangle} \longrightarrow G_{\langle F,P\rangle} \,/\, \mathbb{R}\mathrm{oom}$$

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Substitution systems

$$
\begin{array}{ccccccc}
\langle F,P\rangle & \longmapsto & \mathbb{S}\mathrm{ubst}_{\langle F,P\rangle} & \longrightarrow & G_{\langle F,P\rangle} \,/\, \mathbb{R}\mathrm{oom} & & G_{\langle F,P\rangle} \\
\varphi \big\downarrow & & \Psi_{\varphi} \big\downarrow & \Downarrow \tau_{\varphi} & \big\uparrow \kappa_{\varphi}/\mathbb{R}\mathrm{oom} & & \big\downarrow \kappa_{\varphi} \\
\langle F',P'\rangle & \longmapsto & \mathbb{S}\mathrm{ubst}_{\langle F',P'\rangle} & \longrightarrow & G_{\langle F',P'\rangle} \,/\, \mathbb{R}\mathrm{oom} & & G_{\langle F',P'\rangle}
\end{array}
$$

Quantified Sentences

Consider a generalised substitution system $\mathcal{GS} : \mathbb{S}\mathrm{ig} \to \mathbb{S}\mathrm{ubst}\mathbb{S}\mathrm{ys}$.

Universally-quantified sentence (for a signature Σ):

$$(\forall X)\,\rho$$

where $X \in |\mathbb{S}\mathrm{ubst}_\Sigma|$ and $\rho \in \mathrm{Sen}_\Sigma(X)$.

$M \vDash^{\mathrm{qs}}_\Sigma (\forall X)\,\rho$ if and only if all X-expansions of M satisfy ρ

Existentially-quantified sentences are defined in a similar manner.

Quantified Sentences

Proposition. Every generalised substitution system $\mathcal{GS}\colon \mathbb{S}\mathrm{ig} \to \mathbb{S}\mathrm{ubst}\mathbb{S}\mathrm{ys}$ that has weak model amalgamation, i.e.

$$\begin{array}{ccc} |\mathrm{Mod}(\Sigma)| & \xleftarrow{\ \ -\upharpoonright_{\varphi}\ \ } & |\mathrm{Mod}(\Sigma')| \\ {\scriptstyle -\upharpoonright_{\Sigma}}\uparrow & & \uparrow{\scriptstyle -\upharpoonright_{\Sigma'}} \\ |\mathrm{Mod}_{\Sigma}(X)| & \xleftarrow[\beta_{\varphi,X}]{} & |\mathrm{Mod}_{\Sigma'}(\Psi_{\varphi}(X))| \end{array}$$

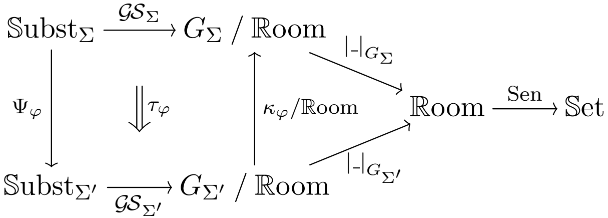
is a weak pullback for every signature morphism $\varphi\colon \Sigma \to \Sigma'$, gives rise to an institution of quantified sentences

$$\mathcal{GS}^{\mathrm{qs}} = \langle \mathbb{S}\mathrm{ig}, \mathrm{QSen}, \mathrm{Mod}, \models^{\mathrm{qs}} \rangle.$$

Local Sentences

Generalised substitution systems $\mathcal{GS}\colon \mathbb{S}\mathrm{ig} \to \mathbb{S}\mathrm{ubst}\mathbb{S}\mathrm{ys}$ determine local-sentence functors $\mathrm{LSen}\colon \mathbb{S}\mathrm{ig} \to [\_ \to \mathbb{S}\mathrm{et}]^\sharp$ that map

- signatures Σ to $\mathcal{GS}_\Sigma \,;\, |\_|_{G_\Sigma} \,;\, \mathrm{Sen}\colon \mathbb{S}\mathrm{ubst}_\Sigma \to \mathbb{S}\mathrm{et}$, and
- signature morphisms $\varphi\colon \Sigma \to \Sigma'$ to $\langle \Psi_\varphi, \tau_\varphi \cdot (|\_|_{G_\Sigma} \,;\, \mathrm{Sen}) \rangle$.



Abstract Logic Programming

Logic-programming framework: $\mathcal{F} = \langle \mathcal{GS}, \mathrm{C}, \mathrm{Q}, \Vdash \rangle$, where

- $\mathcal{GS}$ is a gen. subst. sys. that has weak model amalgamation,
- C and Q are generalised subfunctors of LSen, and
- $\Vdash$ is a generalised subfunctor of $(\mathrm{Q} \times \mathrm{C}) \times \mathrm{Q}$,

such that

1. preservation of queries:

$$N_1 \models_{\Sigma,X} \rho \quad \text{implies} \quad N_2 \models_{\Sigma,X} \rho,$$

   for every $\rho \in \mathrm{Q}_\Sigma(X)$ and $h \colon N_1 \to N_2$ in $\mathrm{Mod}_\Sigma(X)$,

2. soundness of goal-directed rules:

$$\rho_1, \gamma \Vdash_{\Sigma,X} \rho_2 \quad \text{implies} \quad \{\rho_2, \gamma\} \models_{\Sigma,X} \rho_1.$$

Abstract Logic Programming

Example. Relational logic programming is defined over the generalised substitution system $\textsc{qf}\text{-}\mathrm{FOL}^1_{\neq}$ as follows:

- $\mathrm{C}_{\langle F,P\rangle}(X)$ consists of all implications $\bigwedge H \Rightarrow C$, where H is a finite set of relational atoms and C is an atom,
- $\mathrm{Q}_{\langle F,P\rangle}(X)$ consists of all finite conjunctions of atoms $\bigwedge Q$,
- the goal-directed rules are given by

$$\bigwedge(\{C\} \cup Q), \bigwedge H \Rightarrow C \Vdash_{\langle F,P\rangle,X} \bigwedge(Q \cup H).$$

We obtain in this way a logic-programming framework $\mathcal{FOL}^1_{\neq}$.

Abstract Logic Programming

Example. Equational logic programming is defined over the generalised substitution system $\text{QF-FOL}_=$ as follows:

- $\mathrm{C}_{\langle S,F\rangle}(X)$ consists of implications $\bigwedge H \Rightarrow (l = r)$, where H is a finite set of equational atoms and $l = r$ is an atom,
- $\mathrm{Q}_{\langle F,P\rangle}(X)$ consists of finite conjunctions of atoms $\bigwedge Q$,
- the goal-directed rules are given by

$$\bigwedge(\{c[l] = t\} \cup Q), \bigwedge H \Rightarrow (l = r) \Vdash_{\langle S,F\rangle,X} \bigwedge(\{c[r] = t\} \cup Q \cup H)$$

or

$$\bigwedge(\{t = c[l]\} \cup Q), \bigwedge H \Rightarrow (l = r) \Vdash_{\langle S,F\rangle,X} \bigwedge(\{t = c[r]\} \cup Q \cup H).$$

This determines a logic-programming framework $\mathcal{FOL}_=$.

Abstract Logic Programming

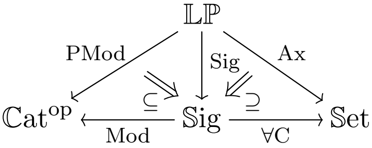
Given a signature Σ in a framework $\mathcal{F} = \langle \mathcal{GS}, \mathrm{C}, \mathrm{Q}, \Vdash \rangle$,

Σ-clause: quantified sentence $(\forall X)\,\gamma$ over Σ such that $\gamma \in \mathrm{C}_\Sigma(X)$.

Σ-query: quantified sentence $(\exists X)\,\rho$ over Σ such that $\rho \in \mathrm{Q}_\Sigma(X)$.

This yields two subfunctors $\forall\mathrm{C}, \exists\mathrm{Q} \subseteq \mathrm{QSen}$.

Logic-programming language $\mathcal{L} = \langle \mathbb{LP}, \mathrm{Sig}, \mathrm{PMod}, \mathrm{Ax} \rangle$ over $\mathcal{F}$:



- a category $\mathbb{LP}$ of logic programs,
- a signature functor $\mathrm{Sig} \colon \mathbb{LP} \to \mathbb{S}\mathrm{ig}$, and
- subfunctors $\mathrm{PMod} \subseteq \mathrm{Sig}\,;\mathrm{Mod}$ and $\mathrm{Ax} \subseteq \mathrm{Sig}\,;\forall\mathrm{C}$,

such that for every program P and model M of P, $M \models^{\mathrm{qs}} \mathrm{Ax}(P)$.

Abstract Logic Programming

Consider a logic program $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle$. A Σ-substitution $\psi\colon X \to Y$ is a $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle$-solution to a Σ-query $(\exists X)\,\rho$ if

- Y is conservative, i.e. $\beta_{\Sigma,Y}\colon \mathrm{Mod}_\Sigma(Y) \to \mathrm{Mod}(\Sigma)$ is surjective on objects, and
- $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle \models^{\mathrm{lp}}_{\Sigma} (\forall Y)\,\psi(\rho)$.

Theorem (Herbrand's theorem). For every program $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle$ and Σ-query $(\exists X)\,\rho$ such that $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle$ has an X-reachable initial model $0_{\langle\!\langle \Sigma, \Gamma \rangle\!\rangle}$, the following statements are equivalent:

1. $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle \models^{\mathrm{lp}}_{\Sigma} (\exists X)\,\rho$,
2. $0_{\langle\!\langle \Sigma, \Gamma \rangle\!\rangle} \models^{\mathrm{qs}}_{\Sigma} (\exists X)\,\rho$,
3. $(\exists X)\,\rho$ admits a $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle$-solution.

Operational Semantics

- $\texttt{add}(0, M, M) \xleftarrow[M]{}$
- $\texttt{add}(\texttt{s}\, M, N, \texttt{s}\, P) \xleftarrow[M,N,P]{} \texttt{add}(M, N, P)$

$$\xmapsto[X_1]{} \texttt{add}(\texttt{s}\, 0, \texttt{s}\, 0, X_1)$$

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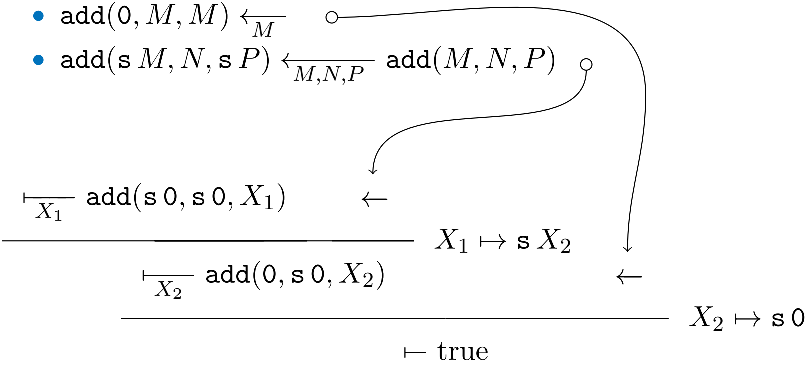

$$X_1 \mapsto \texttt{s}\, X_2 \mapsto \texttt{s}\,\texttt{s}\,0$$

Operational Semantics

- $0 = 0 \leftarrow$
- $0 + M = M \xleftarrow[M\,:\,\texttt{Nat}]{}$
- $(\texttt{s}\, M) + N = \texttt{s}\,(M + N) \xleftarrow[M,N\,:\,\texttt{Nat}]{}$

$$\vdash_{X_1\,:\,\texttt{Nat}} \texttt{s}\,0 + \texttt{s}\,0 = X_1$$

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$$\overline{}\; X_1 \mapsto X_2$$

$$\vdash_{X_2:\,\mathtt{Nat}} \mathtt{s}\,(0 + \mathtt{s}\,0) = X_2 \quad \leftarrow$$

$$\overline{}\; X_2 \mapsto X_3$$

$$\vdash_{X_3:\,\mathtt{Nat}} \mathtt{s}\,\mathtt{s}\,0 = X_3$$

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$$X_2 \mapsto X_3$$

$$\vdash_{X_3\,:\,\mathtt{Nat}} \mathtt{s}\,\mathtt{s}\,0 = X_3 \quad \leftarrow$$

$$\vdash \mathtt{s}\,\mathtt{s}\,0 = \mathtt{s}\,\mathtt{s}\,0 \qquad X_3 \mapsto \mathtt{s}\,\mathtt{s}\,0$$

Operational Semantics

- $0 = 0 \leftarrow$
- $0 + M = M \xleftarrow[M : \mathtt{Nat}]{}$
- $(\mathsf{s}\, M) + N = \mathsf{s}\,(M + N) \xleftarrow[M,N : \mathtt{Nat}]{}$

$$\vdash_{X_1 : \mathtt{Nat}} \mathsf{s}\,0 + \mathsf{s}\,0 = X_1 \quad \leftarrow$$

$$X_1 \mapsto X_2$$

$$\vdash_{X_2 : \mathtt{Nat}} \mathsf{s}\,(0 + \mathsf{s}\,0) = X_2 \quad \leftarrow$$

$$X_2 \mapsto X_3$$

$$\vdash_{X_3 : \mathtt{Nat}} \mathsf{s}\,\mathsf{s}\,0 = X_3 \quad \leftarrow$$

$$X_3 \mapsto \mathsf{s}\,\mathsf{s}\,0$$

$$\vdash \mathsf{s}\,\mathsf{s}\,0 = \mathsf{s}\,\mathsf{s}\,0$$

$$\boxed{X_1 \mapsto X_2 \mapsto X_3 \mapsto \mathsf{s}\,\mathsf{s}\,0}$$

Operational Semantics

Resolution: Let $(\exists X_1)\,\rho_1$ be a query and $(\forall Y_1)\,\gamma_1$ a clause over Σ. A query $(\exists X_2)\,\rho_2$ is derived by resolution from $(\exists X_1)\,\rho_1$ and $(\forall Y_1)\,\gamma_1$ using the computed substitution $\theta_1 \colon X_1 \to X_2$ if there exists a substitution $\psi_1 \colon Y_1 \to X_2$ such that

$$\theta_1(\rho_1), \psi_1(\gamma_1) \Vdash_{\Sigma, X_2} \rho_2.$$

Proposition. For every step $(\exists X_1)\,\rho_1 \longrightarrow\!\!\!\!\!\rightarrow_{\Gamma,\theta_1} (\exists X_2)\,\rho_2$ and every solution $\psi \colon X_2 \to Y$ to the query $(\exists X_2)\,\rho_2$, the substitution $\theta_1\,;\psi$ is a solution to $(\exists X_1)\,\rho_1$.

Operational Semantics

Trivial query: a query $(\exists Y)\,\top$ such that

- Y is conservative and
- any Y-model satisfies $\top$.

Computed answer: (to a query $(\exists X)\,\rho$, with respect to $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle$)
a substitution $\theta \colon X \to Y$ such that

$$(\exists X)\,\rho \twoheadrightarrow^{*}_{\Gamma,\theta} (\exists Y)\,\top$$

for some trivial query $(\exists Y)\,\top$.

Theorem (Soundness of resolution).
For any program $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle$ and any Σ-query $(\exists X)\,\rho$,
every computed answer to $(\exists X)\,\rho$ is a solution to $(\exists X)\,\rho$.

Operational Semantics

Query-completeness: $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle$ is query-complete if for every conservative signature of Σ-variables X, and every X-query ρ,

$$\langle\!\langle \Sigma, \Gamma \rangle\!\rangle \vDash^{\mathrm{lp}}_{\Sigma} (\forall X)\, \rho \quad \text{implies} \quad \rho, X(\Gamma) \Vdash^{*}_{\Sigma,X} \top$$

for some trivial X-query $\top$, where

$$X(\Gamma) = \big\{ \psi(\gamma) \in \mathrm{Sen}_{\Sigma}(X) \;\big|\; (\forall Y)\, \gamma \in \Gamma \text{ and } \psi \colon Y \to X \big\}.$$

Identity clause: (of a query $(\exists X)\, \rho$) a clause $(\forall X)\, \gamma$ such that γ is a tautology and $\rho, \gamma \Vdash_{\Sigma,X} \rho$.

Theorem (Completeness of resolution).
If $\langle\!\langle \Sigma, \Gamma \rangle\!\rangle$ is a query-complete and $(\exists X)\, \rho$ admits an identity $(\forall X)\, \gamma \in \Gamma$ then every solution to $(\exists X)\, \rho$ can be computed.

Conclusions

Summary

- axiomatic theory of logic programming
- properties related to the satisfaction of quantified sentences
- a generalisation of Herbrand's theorem to abstract logic-programming languages
- a sound and (conditionally) complete procedure – but not complete with respect to the implementation – for computing solutions to queries

Outlook

- other forms of logic programming (higher-order, behavioural, service-oriented)
- preservation and reflection of answers along morphisms
- maps of logic-programming languages

Thank you!

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