Constraint Relationships

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Preferences in Constraint Solving

Constraint problem $((X, D), C)$ over constraint domain (X, D)

- variables X, domains $D = (D_x)_{x \in X}$, constraints C

In practice: over-constrained problems

- Which valuations $v \in [X \to D]$ should be preferred?

Existing formalisms:

- Soft constraints (c-semirings, valued constraints)
  - quantify satisfaction (or violation) degree of a valuation
  - hard to capture; normalisation with different scales necessary
- Conditional preference networks (CP-nets)
  - capture preferences over domains w. r. t. other variables' assignments
  - worst case: $\prod_{1 \leq i \leq n} |D_i|$ total orders have to be specified
- Constraint hierarchies
  - categorise constraints on hierarchy levels
  - combining different hierarchies difficult

Preferences in Constraint Solving: Example

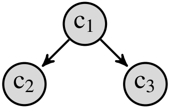
Constraint problem $((\{x, y, z\}, D_x = D_y = D_z = \{1, 2, 3\}), \{c_1, c_2, c_3\})$ with

$$c_1 : x + 1 = y$$
$$c_2 : z = y + 2$$
$$c_3 : x + y \leq 3$$

- Not all constraints can be satisfied simultaneously
  - e. g., c_2 forces z to be 3 and y to be 1, conflicting with c_1
- We can choose between valuations satisfying $\{c_1, c_3\}$ or $\{c_2, c_3\}$.

Constraint relationships

- directly specify which constraints are more important to be satisfied than others, e. g.
  - c_1 more important than c_2
  - c_1 more important than c_3



Constraint Relationships

Approach

- Define acyclic relation R over constraints C
  - more important constraints "above" less important constraints
- Lift this relation to violation sets of valuations
  - dominance properties p

Benefits

- Qualitative formalism — simple to specify, keeps underlying rationale
- "Unbiased" combination of several (disjoint) constraint relationships

A. Schiendorfer, J.-Ph. Steghöfer, A. Knapp, F. Nafz, W. Reif (2013)

Lifting via Dominance Properties

For constraint relationship (C, R) for constraint problem $((X, D), C)$ define valuation $v \in [X \to D]$ to be better than solution $w \in [X \to D]$ w. r. t. relation R and dominance property p if the set of constraints violated by v can be (transitively) worsened to the set of constraints violated by w

$$v >^p_R w \iff \{c \in C \mid v \not\models c\} \, (\rightsquigarrow^p_R)^+ \, \{c \in C \mid w \not\models c\}$$

General assumptions for V, W sets of violated constraints

- Violating one more constraint is worse

$$V \rightsquigarrow^p_R V \uplus \{c\}$$

- Independence of worsenings

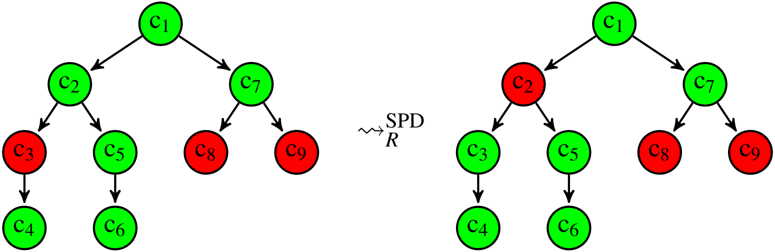
$$\frac{V_1 \rightsquigarrow^p_R W_1 \quad V_2 \rightsquigarrow^p_R W_2}{V_1 \uplus V_2 \rightsquigarrow^p_R W_1 \uplus W_2}$$

Additional rules for each dominance property p

Single-Predecessor Dominance

$$V \uplus \{c\} \rightsquigarrow_R^{\mathrm{SPD}} V \uplus \{c'\} \quad \text{if } c \prec_R c'$$

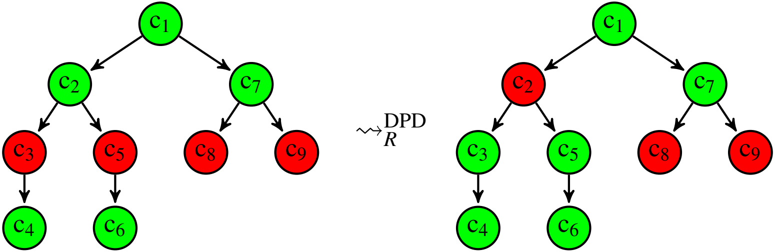
- Example



Direct-Predecessors Dominance

$$V \uplus \{c_1, \ldots, c_k\} \leadsto_R^{\mathrm{DPD}} V \uplus \{c'\} \quad \text{if } \forall c \in \{c_1, \ldots, c_k\} \,.\, c \prec_R c'$$

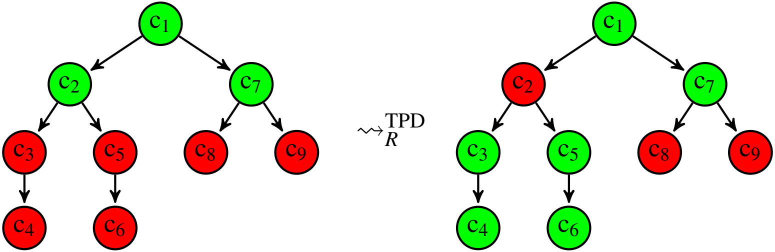
- Example



Transitive-Predecessors Dominance

$$V \uplus \{c_1, \ldots, c_k\} \rightsquigarrow_R^{\mathrm{TPD}} V \uplus \{c'\} \quad \text{if } \forall c \in \{c_1, \ldots, c_k\} \,.\, c \prec_{R^+} c'$$

- Example



Connection to Other Constraint Preference Formalisms

Constraint hierarchies

- can be read as constraint relationships (for “locally predicate better”)
- some constraint relationships not expressible in constraint hierarchies

Conditional preference networks

- “better-as” relation on assignments not necessarily acyclic
- some constraint relationships not expressible in CP-nets

Soft constraints

- translation of constraint relationships into k-weighted CSPs
  - c-semiring, but induces total order on valuations
- More general representation avoiding totalisation?

C-Semirings

C-semiring $A = (|A|, \oplus_A, \otimes_A, \mathbf{0}_A, \mathbf{1}_A)$

- $\oplus_A$ associative, commutative, $\mathbf{1}_A$ annihilator, $\mathbf{0}_A$ neutral element
- $\otimes_A$ associative, commutative, $\mathbf{0}_A$ annihilator, $\mathbf{1}_A$ neutral element
- $\otimes_A$ distributes over $\oplus_A$

C-semiring morphism $\varphi : A \to B$ algebraic homomorphism

In any c-semiring A, $\oplus_A$ is idempotent, inducing a partial order $\preceq_A$

- $a \preceq_A b \iff a \oplus_A b = b$ ("b better than a")
- which is monotone w. r. t. $\oplus_A$ and $\otimes_A$

Intuition: $|A|$ gradings, e. g., for constraint satisfaction/violation

- $\otimes_A$ for combining gradings
- $\oplus_A$ for comparing gradings (typically through $\preceq_A$)
  - $\mathbf{0}_A \preceq_A a \preceq_A \mathbf{1}_A$ for all $a \in |A|$

S. Bistarelli, U. Montanari, F. Rossi (1995)

C-Semirings: Solution Degrees

Soft constraint over constraint domain (X, D) and c-semiring A

- $\gamma : [X \to D] \to |A|$ (grading each valuation)
  - "hard" constraints correspond to soft constraints over initial c-semiring $\mathrm{B} = (\{\bot, \top\}, \vee, \wedge, \bot, \top)$

Solution degree of a valuation for finite set Γ of soft constraints over (X, D), A

- $\Gamma(v) = \bigotimes_A \{\gamma(v) \mid \gamma \in \Gamma\}$

Solution degree comparison of valuations for Γ

- $w \leq_\Gamma v \iff \Gamma(w) \preceq_A \Gamma(v)$ ("v better than w")

Maximum solution degrees of Γ

- $\Gamma^* = \mathrm{Max}^{\preceq_A}\{\Gamma(v) \mid v \in [X \to D]\}$

Optimum solution degree of Γ for finite X and D

- $\mathrm{osd}(\Gamma) = \bigoplus_A \{\Gamma(v) \mid v \in [X \to D]\}$

From Constraint Relationships to C-Semirings (1)

k-weighted CSP

- c-semiring $W_k = (\{0, \dots, k\}, \min, +^k, k, 0)$ with $x +^k y = \min\{k, x + y\}$

Soft constraints over $W_{k_R^p}$ from constraint relationship (C, R)

- recursive assignment of weights according to dominance property p

$$w_R^{\mathrm{SPD}}(c) = 1 + \max\{w_R^{\mathrm{SPD}}(c') \mid c' \in C,\ c' \prec_R c\}$$
$$w_R^{\mathrm{DPD}}(c) = 1 + \sum\{w_R^{\mathrm{DPD}}(c') \mid c' \in C,\ c' \prec_R c\}$$
$$w_R^{\mathrm{TPD}}(c) = 1 + \sum\{2 \cdot w_R^{\mathrm{TPD}}(c') - 1 \mid c' \in C,\ c' \prec_R c\}$$

- $k_R^p = 1 + \sum_{c \in C} w_R^p(c)$

Then

$v >_R^p w$ implies $\sum\{w_R^p(c) \mid c \in C,\ v \not\models c\} < \sum\{w_R^p(c) \mid c \in C,\ w \not\models c\}$

From Constraint Relationships to C-Semirings (2)

Single-predecessor dominance rules for constraint relationship (C, R)

- $V \leadsto_R^{\mathrm{SPD}} V \uplus \{c\}$
- $V \uplus \{c\} \leadsto_R^{\mathrm{SPD}} V \uplus \{c'\}$ if $c \prec_R c'$

reminiscent of Hoare pre-ordering for powerdomains w. r. t. partial order (C, R^*)

- $V \subseteq_{R^*} W \iff \forall c \in V \,.\, \exists c' \in W \,.\, c \preceq_{R^*} c'$

However,

- $\{c_2\} \subseteq_{R^*} \{c_1\}$ would imply $\{c_1, c_2\} \subseteq_{R^*} \{c_1\}$
- additionally, $\{c_1\} \subseteq_{R^*} \{c_1, c_2\}$

But $\otimes_A$ for c-semiring A not idempotent, in general

Constructing a lifting of a partial order to a c-semiring in two steps

- lifting of partial orders to "meet monoids", corresponding to powerdomain construction
- lifting of meet monoids to c-semirings

Partially Ordered, Commutative Monoids

Partially ordered, commutative (poc) monoid $M = (|M|, \cdot_M, \varepsilon_M, \leq_M)$

- $\cdot_M$ associative and commutative, ε_M neutral element
- $\leq_M$ partial order which is monotone w. r. t. $\cdot_M$

Poc monoid morphism $\varphi : M \to N$ algebraic, monotone

M meet (join) monoid if (equivalently)

- $m \cdot_M n \leq_M m$ ($m \leq_M m \cdot_M n$) for all $m, n \in |M|$
- ε_M greatest (smallest) element w. r. t. $\leq_M$

Similar: M. Hölzl, M. Meier, M. Wirsing (2006); F. Gadducci, M. Hölzl, G. V. Monreale, M. Wirsing (2013)

Functor $mMon : \mathrm{cSRng} \to \mathrm{mMon}$ from c-semirings to meet monoids

$$mMon(A) = (|A|, \otimes_A, \mathbf{1}_A, \preceq_A) \qquad mMon(\varphi : A \to B) = \varphi$$

Functor $PO : \mathrm{mMon} \to \mathrm{PO}$ from meet monoids to partial orders

$$PO(M) = (|M|, \leq_M) \qquad PO(\varphi : M \to N) = \varphi$$

From Partial Orders to Poc Monoids

Given partial order P, consider finite multisets $\mathcal{M}_{\text{fin}}\,|P|$ over set $|P|$

- Lower or Hoare ordering $\sqsubseteq_P \subseteq (\mathcal{M}_{\text{fin}}\,|P|) \times (\mathcal{M}_{\text{fin}}\,|P|)$

  $T \sqsubseteq U$ implies $T \sqsubseteq_P U$

  $p \leq_P q$ implies $T \uplus \wr p \int \sqsubseteq_P T \uplus \wr q \int$

  "each element of T can be paired off with a dominating element of U"

- Upper or Smyth ordering $\sqsubseteq^P \subseteq (\mathcal{M}_{\text{fin}}\,|P|) \times (\mathcal{M}_{\text{fin}}\,|P|)$

  $T \sqsubseteq^P U \iff U \sqsubseteq_{P^{-1}} T$

  "each element of U can be paired off with a dominated element of T"

- Convex or Plotkin ordering $\sqsubseteq_P \cap \sqsubseteq^P$

Multiset union monotone w. r. t. $\sqsubseteq_P$, $\sqsubseteq^P$, $\sqsubseteq_P \cap \sqsubseteq^P$

$mMon\langle - \rangle \dashv PO$

For upper ordering over P

- $T \uplus U \Subset^P T$
- $\wr\int$ greatest element for $\Subset^P$

Functor $mMon\langle - \rangle : \mathrm{PO} \to \mathrm{mMon}$ from partial orders to meet monoids

$$mMon\langle P \rangle = (\mathcal{M}_{\mathrm{fin}}\,|P|, \uplus, \wr\int, \Subset^P)$$

$$mMon\langle \varphi : P \to Q \rangle = \lambda \wr p_1, \ldots, p_k \int \in \mathcal{M}_{\mathrm{fin}}\,|P| \,.\, \wr \varphi(p_1), \ldots, \varphi(p_k) \int$$

Unit $\eta_P^{\mathrm{mMon}} : P \to PO(mMon\langle P \rangle)$

$$\eta_P^{\mathrm{mMon}}(p) = \wr p \int$$

Lifting $\varphi^{\sharp_{\mathrm{mMon}}} : mMon\langle P \rangle \to M$ for $\varphi : P \to PO(M)$

$$\varphi^{\sharp_{\mathrm{mMon}}}(\wr p_1, \ldots, p_k \int) = \varphi(p_1) \cdot_M \ldots \cdot_M \varphi(p_k)$$

$cSRng\langle - \rangle \dashv mMon$ (1)

Given a meet monoid M, consider finite sets of pairwise $\leq_M$-incomparable elements $\mathcal{I}_{\text{fin}}(M)$ of $|M|$; define $\tilde{\cup}_M, \tilde{\cdot}_M : \mathcal{I}_{\text{fin}}(M) \times \mathcal{I}_{\text{fin}}(M) \to \mathcal{I}_{\text{fin}}(M)$

- $I \mathbin{\tilde{\cup}_M} J = \mathrm{Max}^{\leq_M}(I \cup J)$
- $I \mathbin{\tilde{\cdot}_M} J = \mathrm{Max}^{\leq_M}\{m \cdot_M n \mid m \in I,\ n \in J\}$

Functor $cSRng\langle - \rangle : \mathrm{mMon} \to \mathrm{cSRng}$ from meet monoids to c-semirings

$$cSRng\langle M \rangle = (\mathcal{I}_{\text{fin}}(M), \tilde{\cup}_M, \tilde{\cdot}_M, \emptyset, \{\varepsilon_M\})$$

$$cSRng\langle \varphi : M \to N \rangle = \lambda\{m_1, \dots, m_k\} \in \mathcal{I}_{\text{fin}}(M)\,.\ \mathrm{Max}^{\leq_N}\{\varphi(m_1), \dots, \varphi(m_k)\}$$

Unit $\eta_M^{\mathrm{cSRng}} : M \to mMon(cSRng\langle M \rangle)$

$$\eta_M^{\mathrm{cSRng}}(m) = \{m\}$$

Lifting $\varphi^{\sharp_{\mathrm{cSRng}}} : cSRng\langle M \rangle \to A$ for $\varphi : M \to mMon(A)$

$$\varphi^{\sharp_{\mathrm{cSRng}}}(\{m_1, \dots, m_k\}) = \varphi(m_1) \oplus_A \cdots \oplus_A \varphi(m_k)$$

$cSRng\langle - \rangle \dashv mMon$ (2)

Hoare ordering in $cSRng\langle M \rangle$ for meet monoid M

$$\begin{aligned} I \preceq_{cSRng\langle M \rangle} J &\iff I \mathbin{\tilde{\cup}_M} J = J \\ &\iff \mathrm{Max}^{\leq_M}(I \cup J) = J \\ &\iff I \subseteq_{PO(M)} J \end{aligned}$$

For specifying soft constraints $\Gamma \subseteq ([X \to D] \to |cSRng\langle M \rangle|)$, it suffices to specify $\mathrm{M} \subseteq ([X \to D] \to |M|)$, setting $\Gamma = \mathrm{M} \circ \eta_M^{\mathrm{cSRng}}$

$$\begin{aligned} \Gamma(v) &= \textstyle\bigotimes_{cSRng\langle M \rangle} \{\gamma(v) \mid \gamma \in \Gamma\} \\ &= \textstyle\bigotimes_{cSRng\langle M \rangle} \{\mu(\eta_M^{\mathrm{cSRng}}(v)) \mid \mu \in \mathrm{M}\} \\ &= \textstyle\bigotimes_{cSRng\langle M \rangle} \{\{\mu(v)\} \mid \mu \in \mathrm{M}\} \\ &= \{\textstyle\prod_M \{\mu(v) \mid \mu \in \mathrm{M}\}\} \\ &= \{\mathrm{M}(v)\} \end{aligned}$$

Connection to Constraint Relationships

Given constraint problem $((X, D), C)$ with constraint relationship (C, R)

Consider

- partial order $P = (C, R^*)$
- meet monoid $M = mMon\langle P^{-1}\rangle$ with grading of $v \in [X \to D]$

  $\mathrm{M}(v) = \prod_M \{c_M(v) \mid c \in C\}$ where $c_M(v) = (v \not\models c \supset \wr c \int ; \wr \int)$

Then

$$\begin{aligned} \mathrm{M}(w) \leq_M \mathrm{M}(v) &\iff \textstyle\prod_M \{c_M(w) \mid c \in C\} \leq_M \prod_M \{c_M(v) \mid c \in C\} \\ &\iff \wr c \mid c \in C,\ w \not\models c \int \Subset^{(R^*)^{-1}} \wr c \mid c \in C,\ v \not\models c \int \\ &\iff \wr c \mid c \in C,\ v \not\models c \int \Subset_{R^*} \wr c \mid c \in C,\ w \not\models c \int \end{aligned}$$

- Corresponding to SPD (on sets of violated constraints)

Branch & Bound (1)

Computation of maximum solution degrees

- soft constraints $\mathrm{M} \subseteq ([X \to D] \to |M|)$ over meet monoid M
- examining partial valuations $p \in [X \to D^?]$
  - $p\!\uparrow$ set of totalisations for p
- trying to cut off non-contributing partial valuations early

$(\alpha, \zeta) \in ([X \to D^?] \to |M|)^2$ tight bounding pair for M if for all $p \in [X \to D^?]$

- $\mathrm{M}(v) \leq_M \zeta(p)$ for all $v \in p\!\uparrow$ (upper bound)
- $\alpha(p) \leq_M \mathrm{M}(v)$ for some $v \in p\!\uparrow$ (lower bound)

Branch & Bound (2)

Assume: – (X, D) finite constraint domain, M meet monoid
– $\mathrm{M} \subseteq ([X \to D] \to |M|)$ finite set of soft constraints
– (α, ζ) tight bounding pair for M

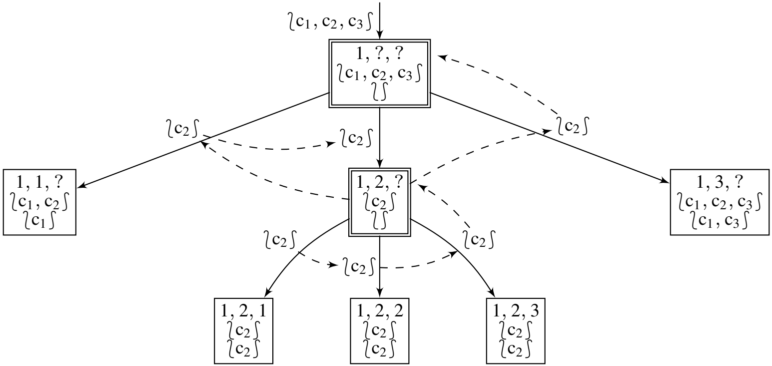
In: – $p \in [X \to D^{?}]$ partial valuation for (X, D)
– $L \subseteq |M|$ finite and pairwise incomparable w.r.t. $\leq_M$

Return: $\mathrm{Max}^{\leq_M}(L \cup \mathrm{M}(p\uparrow)) = L \oplus_{cSRng\langle M\rangle} \mathrm{Max}^{\leq_M} \mathrm{M}(p\uparrow)$

maxSolDegs$_{(\alpha,\zeta)}(p, L) \equiv$
if $\forall x \in X \,.\, p(x) \neq ?$
then return $\mathrm{Max}^{\leq_M}(L \cup \{\zeta(p)\})$ **fi**
$x \leftarrow$ **choose** $\{x \in X \mid p(x) = ?\}$
$L \leftarrow \mathrm{Max}^{\leq_M}(L \cup \{\alpha(p\{x \mapsto d\}) \mid d \in D_x\})$
for $d \in D_x$
do if $\neg\exists l \in L \,.\, \zeta(p\{x \mapsto d\}) \leq_M l$
then $L \leftarrow$ maxSolDegs$_{(\alpha,\zeta)}(p\{x \mapsto d\}, L)$ **fi od**
return L

Branch & Bound: Example

- $(X, D) = (\{x, y, z\}, D_x = D_y = D_z = \{1, 2, 3\})$
- $(C, R) = (\{c_1 : x + 1 = y,\ c_2 : z = y + 2,\ c_3 : x + y \le 3\}, \{c_1 \succ c_2,\ c_3\})$
- $M = mMon\langle (C, (R^*)^{-1}) \rangle$
- $\alpha(p) = \wr c \in C \mid sc(c) \subseteq def(p),\ p \not\models c \int \uplus \wr c \in C \mid sc(c) \not\subseteq def(p) \int$
- $\zeta(p) = \wr c \in C \mid sc(c) \subseteq def(p),\ p \not\models c \int$



Conclusions and Future Work

Constraint relationships

- relation to c-semirings
- using meet monoids

Future work

- search heuristics
- conditional statements
- learning/abductive reasoning for preference elicitation
- applications, e.g., optimisation in power management systems