Towards the Coalgebraic Guarded Fragment

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- The guarded fragment of FOL:
  - Decidable (in EXPTIME if the arity of predicates is bounded)
  - Contains the modal fragment
  - Characterized by guarded bisimulation
- Here: analyse the guarded fragment modally
- Generalize this to the coalgebraic setting
- Partly speculative

Coalgebraic Modalities on one Slide

Kripke semantics, coalgebraically:

- $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ has coalgebras $\xi : X \to \mathcal{P}(X)$ = Kripke frames
- Putting

  $$[\![\Box]\!]_X : 2^X \to 2^{\mathcal{P}(X)}, \quad A \mapsto \{B \in \mathcal{P}(X) \mid B \subseteq A\}$$

  have

  $$x \models_\xi \Box\phi \quad \text{iff} \quad \xi(x) \in [\![\Box]\!]_X(\{y \mid y \models_\xi \phi\})$$

Generalize this:

- Modal similarity type Λ
- Models are coalgebras $\xi : X \to TX$ for functor $T : \mathbf{Set} \to \mathbf{Set}$
- Interpret $\heartsuit \in \Lambda$ by $[\![\heartsuit]\!]_X : 2^X \to 2^{TX}$

Examples:

- Multiset functor $\mathcal{B}_{\mathbb{N}}$, multigraphs $\bullet \xrightarrow{n} \bullet$, graded modalities $\Diamond_k$
- Distribution functor D, Markov chains, probabilistic modalities L_p
- Conditional logics, coalition logic/alternating-time logic, ...

The Guarded Fragment

Guarded quantification:

$$\forall \vec{y}.\, a(\vec{x},\vec{y}) \to \phi(\vec{x},\vec{y})$$
$$\exists \vec{y}.\, a(\vec{x},\vec{y}) \wedge \phi(\vec{x},\vec{y})$$

with $a(\vec{x},\vec{y})$ atomic.

Here: restrict to modal correspondence language, i.e. at most binary predicates.

$$ST_x(\Box\phi) = \forall y.\, xRy \to ST_y(\phi)$$
$$ST_x(\Diamond\phi) = \exists y.\, xRy \wedge ST_y(\phi)$$

The Guarded Fragment as a Modal Logic

$$\phi, \psi ::= a \mid i \mid \circlearrowleft \mid \overset{\circlearrowleft}{\Diamond} \mid \overset{\circlearrowleft}{\blacklozenge} \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi \mid \blacklozenge\phi \mid \mathsf{A}\phi \mid @_{\circlearrowleft}\phi$$

(As a DL: $\mathcal{ALCHIO}$ + $\{\circlearrowleft, \overset{\circlearrowleft}{\Diamond}, \overset{\circlearrowleft}{\blacklozenge}, @_{\circlearrowleft}\}$.)

Models $M = (X, R, \pi)$ with (X, R) Kripke frame, $\pi : \mathsf{P} \cup \mathsf{N} \to \mathcal{P}(C)$ hybrid valuation.

$$M, c, d \models \circlearrowleft \text{ iff } c = d$$

$$M, c, d \models \overset{\circlearrowleft}{\Diamond} \text{ iff } Rdc$$

$$M, c, d \models \overset{\circlearrowleft}{\blacklozenge} \text{ iff } Rcd$$

$$M, c, d \models \Diamond\phi \text{ iff } \exists e.\, Rde \wedge M, d, e \models \phi$$

$$M, c, d \models \blacklozenge\phi \text{ iff } \exists e.\, Red \wedge M, d, e \models \phi$$

$$M, c, d \models \mathsf{A}\phi \text{ iff } \forall e.\, M, e, e \models \phi$$

$$\text{iff } \forall e.\, e = e \to M, e, e \models \phi$$

$$M, c, d \models @_{\circlearrowleft}\phi \text{ iff } M, c, c \models \phi$$

Examples

(In DL notation)

The narcissist (Marx):

$$\exists \mathsf{loves}. \circlearrowleft$$

The nice guy:

$$\forall \mathsf{knows}^{-1}. (\mathsf{sane} \to \overset{\circlearrowleft}{\exists} \mathsf{likes})$$

What is this coalgebraically?

- $\mathcal{ALCHO}$ with general TBoxes $\Longrightarrow$ Coalgebraic DL (Schröder/Pattinson/Kupke IJCAI 2009)
- $\circlearrowleft$: clear
- $\overset{\circlearrowleft}{\Diamond}$ and $\overset{\circlearrowleft}{\blacklozenge}$: less clear
- Inverses: even less.

Start with the Clear Part

CDL + $\{\circlearrowleft, @_{\circlearrowleft}\}$:

$$\phi ::= \bot \mid i \mid \circlearrowleft \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \heartsuit\phi \mid @_i\phi \mid @_{\circlearrowleft}\phi \qquad (\heartsuit \in \Lambda)$$

Prove FMP/EXPTIME via global caching:

- ϕ satisfiable, Σ closure of ϕ under subformulas, $\neg$, $@$, $\circlearrowleft$-instantiation
- $\Sigma' \subseteq \Sigma$ closed formulas
- $K \subseteq @_N\Sigma'$ $@$-theory
- $S_K =$ max. satisfiable subsets of Σ' above K

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- $K \subseteq @_N\Sigma'$ $@$-theory
- $S_K =$ max. satisfiable subsets of Σ' above K
- S_K does not suffice to interpret $\circlearrowleft$
  - Two different states can have the same Σ'-theory (unless they are named)

Duplicating States

Put $K_i = \{\phi \mid @_i\phi \in K\}$, $S_N = \{K_i \mid i \in \Sigma\}$. Then

$$\bar{S}_K = S_N \cup 2 \times (S_K - S_N); \qquad l : \bar{S}_K \to S_K$$

Pseudoextension $[\rho]_x$ for $x \in \bar{S}_K$:

$$\begin{aligned}
[\circlearrowleft]_x &= \{x\} \\
[@_{\circlearrowleft}\rho]_x &= \{y \in \bar{S}_K \mid x \in [\rho]_x\} \\
[@_i\rho]_x &= \{y \in \bar{S}_K \mid K_i \in [\rho]_{K_i}\} = \{y \in \bar{S}_K \mid \rho(i) \in K_i\} \\
[\heartsuit\rho]_x &= \{y \in \bar{S}_K \mid \heartsuit\rho \in l(y)\}.
\end{aligned}$$

$\to$ Intuitively clear but technically too complex.

Match up with Open Formulas

For $A \subseteq \Sigma$ (open formulas)

$$S_{K,A} = \text{ max. sat. subsets of } \Sigma \text{ over } K \cup @_{\circlearrowleft} A$$

For $x \in \bar{S}_K$, define $r_x : S_{K,A} \to \bar{S}_K$:

$$r_{A,\varepsilon}(B) = \begin{cases} K_i & \text{if } i \in B \text{ for some } i \in N \\ (\varepsilon \leftrightarrow (\circlearrowleft \in B), B \cap \Sigma') & \text{otherwise.} \end{cases}$$

$$r_{K_i}(B) = \begin{cases} K_i & i \in B \\ (\bot, B \cap \Sigma') & \text{otherwise} \end{cases}$$

Key lemma:

$$r_x^{-1}[\rho]_x = \{B \in S_{K,A} \mid \rho \in B\}.$$

Inverses

- Generally: ??
- Seems clear at least for multigraphs:

$$F : a \xrightarrow{n} b \qquad P : b \xrightarrow{n} a$$

- But: this is modally indistinguishable from causally coherent frames

$$F : a \xrightarrow{>0} b \quad \text{iff} \quad P : b \xrightarrow{>0} a$$

- (Aside: we have used this to prove the FMP for $\mathcal{ALCHIQ}$)
- Current summary: inverses are not coalgebraic

Another Look at $\overset{\circlearrowleft}{\Diamond}$

Can see $\overset{\circlearrowleft}{\Diamond}$ as an instance of $\Diamond^*$, where

$$M, c, d \models \Diamond\phi \text{ iff } \exists e.\, Red \wedge M, d, e \models \phi$$
$$M, c, d \models \Diamond^*\phi \text{ iff } \exists e.\, Red \wedge M, c, e \models \phi$$

Then $\overset{\circlearrowleft}{\Diamond} = \Diamond^* \circlearrowleft$.

Can we add $\Diamond^*$ to the language?

I-me

(Marx)

- Single-variable fragment of $\mathcal{H}(@,\downarrow)$:-)
- *I* names the current state, later referred to as *me*.

Examples:

$$\begin{aligned} \text{narcissist} &= I.\, \exists \text{loves}.\, me \\ \text{niceGuy} &= I.\, \forall \text{knows}^{-1}.\, (\text{sane} \to \exists \text{likes}.\, me) \\ \text{stepMother} &= I.\, \exists \text{hasSpouse}.\, \exists \text{hasChild}.\, \forall \text{hasChild}^{-1}.\, \neg me \end{aligned}$$

Encoding I-me

- I distributes over Boolean operations
- $I.\Diamond \leadsto \Diamond$
- $\Diamond \leadsto \Diamond^*$

Unrestricted I-me is undecidable!

Open problem: can we find suitable restrictions on I-me?

That's it.

Sichere Kognitive Systeme

Thanks for your attention!