

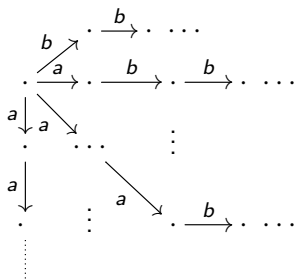
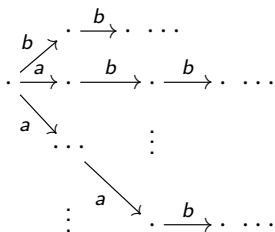
Coalgebraic Traces Revisited

Corina Cîrstea
University of Southampton

Rome, 16 March 2013

Motivation

- well understood theory of **finite traces** [HJS 2007]
 - coinduction in Kleisli category
- some work on **infinite traces** [Jacobs 2004], [Cîrstea 2011] ...
 - applies to non-deterministic and probabilistic systems
- **but problems still remain!**



This Talk

- ① a unified account of finite and maximal (possibly infinite) traces
- ② a fibrational account of maximal traces
 - based on [Hasuo et al 2013]

The General Setting

Similarly to [HJS 2007], we work with $T \circ F$ -coalgebras, where:

- commutative monad $T : C \rightarrow C$ describes the **branching type**
e.g. \mathcal{P} (powerset), \mathcal{S} (sub-probability distributions), \mathcal{M} (finite multisets)
- $F : C \rightarrow C$ describes the **transition type**
e.g. Id , $A \times \text{Id}$, $1 + A \times \text{Id}$
- distributive law $\lambda : F \circ T \Rightarrow T \circ F$ (compatible with monad structure)

This covers:

- non-deterministic systems: $\mathcal{P}(1 + A \times \text{Id})$
- probabilistic systems: $\mathcal{S}(A \times \text{Id})$
- weighted transition systems: $\mathcal{M}(A \times \text{Id})$
- ...

Finite vs Maximal Traces: Our Expectations

Fix $T \circ F$ -coalgebra $\gamma : X \rightarrow TFX$.

Finite traces:

- initial F -algebra (I, α) gives the possible finite traces
- finite trace map $\text{ftr}_\gamma : X \rightarrow T(I)$

Maximal traces:

- final F -coalgebra (Z, ζ) gives the possible infinite traces
- maximal trace map $\text{mtr}_\gamma : X \rightarrow T(Z)$

Both cases:

- $T = \mathcal{P} \Rightarrow$ sets of traces
- $T = \mathcal{S} \Rightarrow$ sub-probability *distributions* over traces
- trace map must "commute" with coalgebra structure ...

Maximal Traces: Expectations Made Formal

- working in $KI(T)$ abstracts away branching structure [HJS 2007] !
- $F : \text{Set} \rightarrow \text{Set}$ lifts to $\bar{F} : KI(T) \rightarrow KI(T)$ (using λ)

$$\begin{array}{ccc} KI(T) & \xrightarrow{\bar{F}} & KI(T) \\ \uparrow & & \uparrow \\ \text{Set} & \xrightarrow{F} & \text{Set} \end{array}$$

- $\gamma : X \rightarrow TFX$ yields $\bar{\gamma} : X \rightarrow \bar{F}X$
- $\zeta : Z \rightarrow FZ$ lifts to $\bar{\zeta} : Z \rightarrow \bar{F}Z$
- trace maps must define \bar{F} -coalgebra morphism:

$$\begin{array}{ccc} X & \xrightarrow{\bar{\gamma}} & \bar{F}X \\ \text{mtr}_\gamma \downarrow & & \downarrow \bar{F}\text{mtr}_\gamma \\ Z & \xrightarrow{\bar{\zeta}} & \bar{F}Z \end{array}$$

From Comm. Monads to Comm. Monoids

[Coumans & Jacobs 2011]

For $T : \mathcal{C} \rightarrow \mathcal{C}$ a commutative monad:

- commutative monoid $(T1, \eta_1(*), \bullet)$
 - unit from $\eta_1 : 1 \rightarrow T(1)$
 - multiplication:

$$T(1) \times T(1) \xrightarrow{st} T(1 \times T(1)) \xrightarrow{T\pi_2} T^2(1) \xrightarrow{\mu_1} T(1)$$

- e.g. $T = \mathcal{P}$: $\{0, 1\}$ with \wedge
- e.g. $T = \mathcal{S}$: $[0, 1]$ with $*$
- e.g. $T = \mathcal{M}$: \mathbb{N} with $*$

From Additive Comm. Monads to Comm. Monoids [Coumans & Jacobs 2011]

For $T : \mathcal{C} \rightarrow \mathcal{C}$ a commutative monad with $T0 = 1$:

- 0 element in TX : $T0 \xrightarrow{T!_X} TX$
 $* \longmapsto 0$

- zero map $0 : Y \rightarrow TX$: $Y \xrightarrow{!_Y} T0 \xrightarrow{T!_X} TX$

- $T(X + Y) \xrightarrow{\langle \mu \circ Tp_1, \mu \circ Tp_2 \rangle} T(X) \times T(Y)$

where $X + Y \xrightarrow{p_1 = [\eta, 0]} TX$

- $\langle \mu \circ Tp_1, \mu \circ Tp_2 \rangle$ iso \implies comm. monoid $(TX, 0, +)$

- e.g. $T = \mathcal{P}$: $\{0, 1\}$ with \vee

- e.g. $T = \mathcal{S}$: not an iso

- e.g. $T = \mathcal{M}$: \mathbb{N} with $+$

From Partially Additive Comm. Monads to Partial Comm. Monoids

- $\langle \mu \circ T\rho_1, \mu \circ T\rho_2 \rangle$ mono \implies partial comm. monoid: $(T(X), 0, +)$:

$$\begin{array}{ccc}
 T(X) & \xleftarrow{[1_X, 1_X]} & T(X + X) & \xrightarrow{\langle \mu \circ T\rho_1, \mu \circ T\rho_2 \rangle} & T(X) \times T(X) \\
 & & & \dashleftarrow{\text{partial}} & \\
 & & & \text{+} &
 \end{array}$$

- e.g. $T = \mathcal{S}$: $[0, 1]$ with partial +

Some Partial Orders

- assume partial comm. monoid $(T(X), 0, +)$
- for $x, y \in TX$:
$$x \sqsubseteq y \text{ iff } \exists z \in TX. x + z = y$$
- above yields partial order with $\perp = 0$ on $T1$
 - instantiates to natural orders for $T = \mathcal{P}$, $T = \mathcal{S}$, $T = \mathcal{M}$
- also yields (pointwise) partial order with \perp on $Kl(T)$ -homsets, in particular on $(X \rightarrow TZ)$
- finally, it yields (pointwise) partial order on $X \rightarrow (Z \rightarrow T1)$

Note: in all our examples, TZ is (up to iso) a subset of $(Z \rightarrow T1)$!

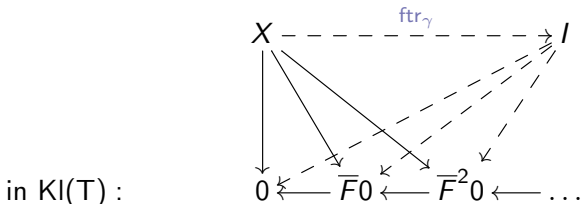
Summary of Examples

- $T = \mathcal{P}$: $(\{0, 1\}, 1, \wedge, 0, +, \leq)$, $\perp = 0$, $\top = 1$
- $T = \mathcal{S}$: $([0, 1], 1, *, 0, +, \leq)$, $\perp = 0$, $\top = 1$
- $T = \mathcal{M}$: $(\mathbb{N}, 1, *, 0, +, \leq)$, $\perp = 0$, **no \top**

Summary of Finite Trace Semantics [HJS 2007]

- initial object in Set gives final object in $\text{Kl}(\mathbb{T})$
- initial F -algebra (I, α) in Set gives final \overline{F} -coalgebra $(I, \overline{\alpha})$ in $\text{Kl}(\mathbb{T})$
- finality in $\text{Coalg}(\overline{F})$ yields finite trace semantics

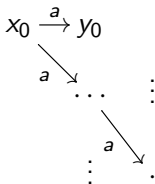
$$\text{ftr}_\gamma : (X, \overline{\gamma}) \rightarrow (I, \overline{\alpha})$$



in Set :

$$0 \longrightarrow F0 \longrightarrow F^2 0 \longrightarrow \dots$$

Example: Finite Traces for LTSs



step 0: $x_0 \mapsto \{\}$, $y_0 \mapsto \{\}$

step 1: $x_0 \mapsto \{\}$, $y_0 \mapsto \{*\}$

step 2: $x_0 \mapsto \{a^*\}$, $y_0 \mapsto \{*\}$

...

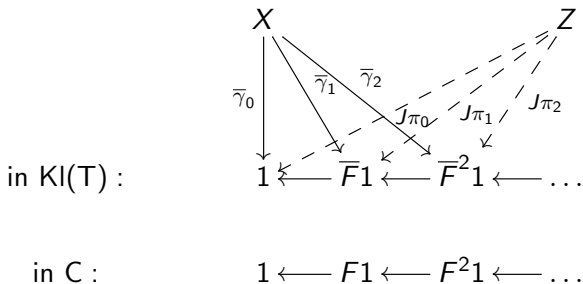
step n: $x_0 \mapsto \{a^*, \dots, a^n*\}$

...

ω : $x_0 \mapsto \{a^n* \mid n = 1, 2, \dots\}$

Maximal Traces – Some Problems

- replace initial F -algebra by final F -coalgebra (Z, ζ)
- viewing $T \circ F$ -coalgebra as \bar{F} -coalgebra $(X, \bar{\gamma})$ yields

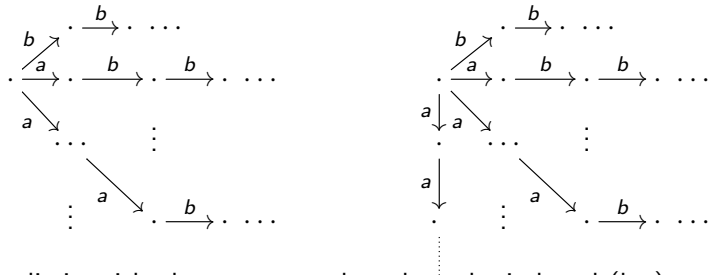


Some problems:

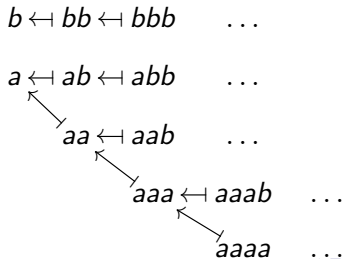
- $(\bar{\gamma}_i)_{i \in \omega}$ only defines a cone when T affine ($\eta_1 : 1 \rightarrow T1$ iso)
- even for affine T , $(Z, (J\pi_i)_{i \in \omega})$ **not always limiting!**

An Example

Two $\mathcal{P}(A \times \text{Id})$ -systems:

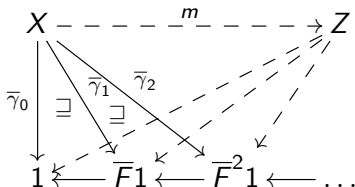


Cannot distinguish above systems based on the induced (lax) cones:



Maximal Traces – Towards a Solution

- work with lax cones:



- works if $\eta_1(*)$ is \top for \sqsubseteq on $T(1)$
- trace map not uniquely determined by $(\bar{\gamma}_i)_{i \in \omega}$!
- many $m : X \rightarrow Z$ s.t. $m; J\pi_i \sqsubseteq \bar{\gamma}_i$ for $i \in \omega$
 - maximal one only defines oplax coalgebra morphism $m; \bar{\zeta} \sqsubseteq \bar{\gamma}; \bar{F}m$
 - ... and is not always the right choice!

Lesson: need to go beyond ω ...

Finite Traces Revisited

- assume $\text{Kl}(T)(X, Z)$ is a poset with \perp and limits of increasing chains
 - true for the powerset, sub-distribution and multiset monads
- monotone operator on $\text{Kl}(T)(X, Z)$:

$$\text{Kl}(T): \quad X \xrightarrow{f} I \quad \xrightarrow{O} \quad X \xrightarrow{\bar{\gamma}} \bar{F}X \xrightarrow{\bar{F}f} \bar{F}I \xrightarrow{\bar{\alpha}} I$$

Definition

Let $\gamma : X \rightarrow TFX$. The **finite trace map** $\text{ftr}_\gamma : X \rightarrow TI$ of γ is the least fixed point of the operator O .

Maximal Traces - a Solution

- no \top in $\text{Kl}(T)(X, Z) \dots$
- so instead of maps $X \rightarrow Z$ in $\text{Kl}(T)$ consider maps $X \rightarrow (T1)^Z$ in Set
- assume **T1-predicate lifting for F** : $\tilde{\lambda}_X : F(T1)^X \rightarrow (T1)^{FX}$
 - think of elements of $(T1)^Y$ as predicates over Y
- assume $T1$ is a poset with \top and limits of decreasing chains
 - true for the powerset and sub-distribution monads
 - extends to poset $(X \rightarrow (T1)^Z)$
- monotone operator \tilde{O} on $(X \rightarrow (T1)^Z)$, mapping $X \xrightarrow{f} (T1)^Z$ to

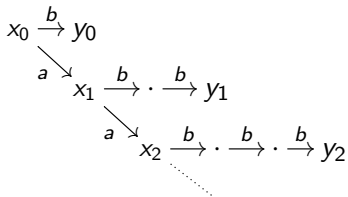
$$X \xrightarrow{\gamma} TFX \xrightarrow{TFf} TF(T1)^Z \xrightarrow{T\tilde{\lambda}_Z} T(T1)^{FZ} \xrightarrow{\alpha} (T1)^{FZ} \xrightarrow{(T1)^\zeta} (T1)^Z$$

Definition

Let $\gamma : X \rightarrow TFX$. The **maximal trace map** $\text{mtr}_\gamma : X \rightarrow (T1)^Z$ of γ is the greatest fixed point of the operator \tilde{O} .

Conjecture: mtr_γ is a map in $\text{Kl}(T)$.

Example: Maximal Traces for LTSs



step 0: $x_i \mapsto \bar{A}, y_i \mapsto \bar{A}$

$$\bar{A} = A^* \cup A^\omega$$

step 1: $x_i \mapsto b\bar{A} \cup a\bar{A}, y_i \mapsto \{*\}$

step 2: $x_0 \mapsto \{b^*\} \cup ab\bar{A} \cup aa\bar{A}$
 $x_{i+1} \mapsto bb\bar{A} \cup ab\bar{A} \cup aa\bar{A}$

...

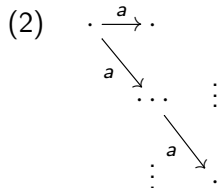
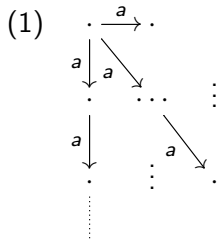
step ω : $x_i \mapsto \{a^j b \mid j \in \mathbb{N}\} \cup \{aa \dots\}$

\Rightarrow all invalid traces (including infinite ones) rejected in finite time!

Is this always the case ?

Example: Maximal Traces for LTSs

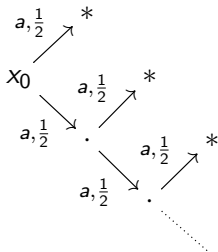
No!



Infinite trace $aa\dots$ not rejected at ω for system (2)!

Question: when can we stop at ω ?

Example: Maximal Traces for PTSs



After ω steps: $x_0 \mapsto (a^* \mapsto \frac{1}{2}, aa^* \mapsto \frac{1}{4}, \dots, a^\omega \mapsto 0)$

Fixpoint always reached at (or before) ω ! Reason: \mathcal{S} preserves limits of ω^{op} -chains.

Coinductive Predicates [Hasuo et al 2013]

- poset fibration: \mathbb{P} (each fibre is a poset)

$$\begin{array}{c} \mathbb{P} \\ p \downarrow \\ \text{Set} \end{array}$$

- predicate lifting: $\mathbb{P} \xrightarrow{\varphi} \mathbb{P}$ (φ preserves cartesian maps)

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\varphi} & \mathbb{P} \\ p \downarrow & & \downarrow p \\ \text{Set} & \xrightarrow{F} & \text{Set} \end{array}$$

- φ -coinductive predicate for $\gamma : X \rightarrow FX$: final $\gamma^* \circ \varphi_X$ -coalgebra

$$\mathbb{P}_X \xrightarrow{\varphi_X} \mathbb{P}_{FX} \xrightarrow{\gamma^*} \mathbb{P}_X$$

- φ -invariant for $\gamma : X \rightarrow FX$: $\gamma^* \circ \varphi_X$ -coalgebra
- e.g. can be used to interpret basic fragment of modal μ -calculus

Maximal Traces from Coind. Predicates

- predicate over X :

$$\begin{array}{c} X \\ q \downarrow \\ (Z \rightarrow T1) \end{array}$$

- predicate lifting for F :

$$\begin{array}{ccc} X & & FX \\ q \downarrow & & \downarrow Fq \\ (Z \rightarrow T1) & & F(Z \rightarrow T1) = FTZ \\ & & \downarrow \lambda_Z \\ & & (FZ \rightarrow T1) = TFZ \\ & & \downarrow -\circ\zeta \\ & & (Z \rightarrow T1) \end{array}$$

- predicate lifting for T :

$$\begin{array}{ccc} Y & & TY \\ q \downarrow & & \downarrow Tq \\ (Z \rightarrow T1) & & T(Z \rightarrow T1) \\ & & \downarrow \alpha \\ & & (Z \rightarrow T1) \end{array}$$

Maximal Traces from Coind. Predicates

For $\gamma : X \rightarrow TFX$, take the maximal trace map to be the $\psi \circ \varphi$ -coinductive predicate in γ :

$$\mathbb{P}_X \xrightarrow{\varphi_X} \mathbb{P}_{FX} \xrightarrow{\psi_{FX}} \mathbb{P}_{TFX} \xrightarrow{\gamma^*} \mathbb{P}_X$$

Theorem (Hasuo et al 2013)

For well-founded p , finitary F , coinductive predicate φ for F and F -coalgebra γ , the fixpoint is reached at ω .

Cannot directly use above result as our fibrations are not well-founded.

Future Work

- further exploit results in [Hasuo et al 2013]
- other monads (e.g. Giry, multiset)
- connection to automata
- trace logics
- ...