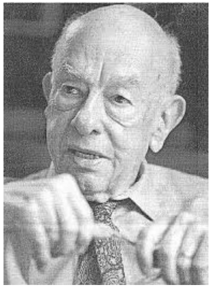
Predicate Functor Logic (Quine 1971)



"Logic in his adolescent phase was algebraic. There was Boole's algebra of classes and Peirce's algebra of relations. But in 1879 logic come of age, with Frege's quantification theory. Here the bound variables, so characteristic of analysis rather than of algebra, became central to logic."

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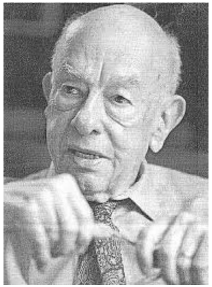

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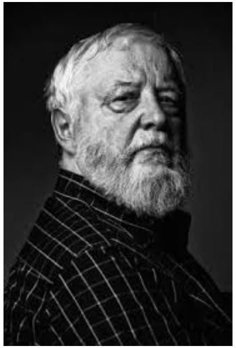


Adjointness in Foundations (Lawvere 1969)

Boole's algebra of classes 1847

Boole's algebra of classes 1847

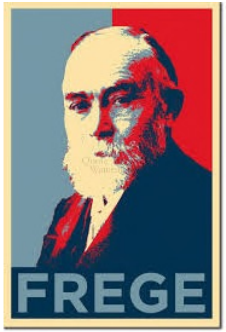
Operations: $(\vee, 0, \wedge, 1, \neg)$

Boole's algebra of classes 1847

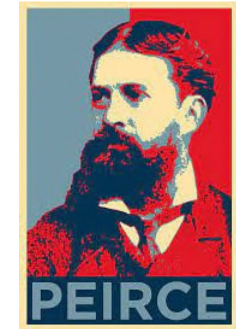
Operations: $(\vee, 0, \wedge, 1, \neg)$

Laws:

$p \vee q = q \vee p$	$p \vee p \wedge q = p$
$p \wedge q = q \wedge p$	$p \wedge (p \vee q) = p$
$(p \vee q) \vee r = p \vee (q \vee r)$	$0 \vee p = p$
$(p \wedge q) \wedge r = p \wedge (q \wedge r)$	$0 \wedge p = 0$
$p \wedge (q \vee r) = p \wedge q \vee p \wedge r$	$1 \vee p = 1$
$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$	$1 \wedge p = p$
$p \vee p = p$	$\neg(p \vee q) = \neg p \wedge \neg q$
$p \wedge p = p$	$\neg(p \wedge q) = \neg p \vee \neg q$



FREGE v.s. PEIRCE



FREGE

1879

Motivated by the pursuit of foundations of mathematics; Inspired by real analisys

Brings the concepts of variables and functions into the logical realm

PEIRCE

1870

Inspired by the work of De Morgan on relational reasoning

Introduces a calculus in which operations allow to combine relations and satisfy a set of algebraic laws

The Calculus of Relations

Beyond the boolean operators, it has

$$R \mathbin{\circ\!\!\!\raisebox{-0.5ex}{;}} S \stackrel{\text{def}}{=} \{(x,z) \mid \exists y \in Y \,.\, (x,y) \in R \wedge (y,z) \in S\} \subseteq X \times Z$$

$$R \mathbin{\bullet\!\!\!\raisebox{-0.5ex}{;}} S \stackrel{\text{def}}{=} \{(x,z) \mid \forall y \in Y \,.\, (x,y) \in R \vee (y,z) \in S\} \subseteq X \times Z$$

$$id^{\circ}_X \stackrel{\text{def}}{=} \{(x,y) \mid x = y\} \subseteq X \times X$$

$$id^{\bullet}_X \stackrel{\text{def}}{=} \{(x,y) \mid x \neq y\} \subseteq X \times X$$

$$R^{\dagger} \stackrel{\text{def}}{=} \{(y,x) \mid (x,y) \in R\}$$

for all sets X, Y and Z

and relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$

The Calculus of Relations

$R \in \Sigma$

Syntax

$$
\begin{aligned}
E \quad ::= \quad & R \;\mid\; id^{\circ} \;\mid\; E \,\mathring{;}\, E \;\mid\; id^{\bullet} \;\mid\; E \,\bullet\!\!;\, E \;\mid \\
& E^{\dagger} \;\mid\; \top \;\mid\; E \cap E \;\mid\; \bot \;\mid\; E \cup E \;\mid\; \overline{E}
\end{aligned}
$$

The Calculus of Relations

$R \in \Sigma$

Syntax

$$
\begin{array}{rcl}
E & ::= & R \mid id^{\circ} \mid E \,\text{⨾}\, E \mid id^{\bullet} \mid E \,\bullet\, E \mid \\
 & & E^{\dagger} \mid \top \mid E \cap E \mid \bot \mid E \cup E \mid \overline{E}
\end{array}
$$

Semantics given $\mathcal{I} = (X, \varrho)$ where X is a set and $\varrho : \Sigma \to \mathcal{P}(X \times X)$

The Calculus of Relations

$R \in \Sigma$

Syntax

$$
\begin{array}{rcl}
E & ::= & R \mid id^\circ \mid E \fatsemi E \mid id^\bullet \mid E \mathbin{;} E \mid \\
 & & E^\dagger \mid \top \mid E \cap E \mid \bot \mid E \cup E \mid \overline{E}
\end{array}
$$

Semantics given $\mathcal{I} = (X, \rho)$ where X is a set and $\rho : \Sigma \to \mathcal{P}(X \times X)$

$$
\begin{array}{lll}
\langle R \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \rho(R) & \langle id^\circ \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} id^\circ_X & \langle E_1 \fatsemi E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \fatsemi \langle E_2 \rangle_{\mathcal{I}} \\
\langle E^\dagger \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E \rangle^\dagger_{\mathcal{I}} & \langle id^\bullet \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} id^\bullet_X & \langle E_1 \mathbin{;} E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \mathbin{;} \langle E_2 \rangle_{\mathcal{I}} \\
\langle \overline{E} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \overline{\langle E \rangle_{\mathcal{I}}} & \langle \bot \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \varnothing & \langle E_1 \cup E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cup \langle E_2 \rangle_{\mathcal{I}} \\
 & \langle \top \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} X \times X & \langle E_1 \cap E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cap \langle E_2 \rangle_{\mathcal{I}}
\end{array}
$$

The Calculus of Relations

Syntax

$R \in \Sigma$

$$E \;::=\; R \mid id^{\circ} \mid E \fatsemi E \mid id^{\bullet} \mid E \mathbin{\bullet} E \mid E^{\dagger} \mid \top \mid E \cap E \mid \bot \mid E \cup E \mid \overline{E}$$

Semantics

given $\mathcal{I} = (X, \rho)$ where X is a set and $\rho : \Sigma \to \wp(X \times X)$

$$\langle R \rangle_{\mathcal{I}} \overset{\text{def}}{=} \rho(R) \qquad \langle id^{\circ} \rangle_{\mathcal{I}} \overset{\text{def}}{=} id^{\circ}_{X} \qquad \langle E_1 \fatsemi E_2 \rangle_{\mathcal{I}} \overset{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \fatsemi \langle E_2 \rangle_{\mathcal{I}}$$

$$\langle E^{\dagger} \rangle_{\mathcal{I}} \overset{\text{def}}{=} \langle E \rangle_{\mathcal{I}}^{\dagger} \qquad \langle id^{\bullet} \rangle_{\mathcal{I}} \overset{\text{def}}{=} id^{\bullet}_{X} \qquad \langle E_1 \mathbin{\bullet} E_2 \rangle_{\mathcal{I}} \overset{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \mathbin{\bullet} \langle E_2 \rangle_{\mathcal{I}}$$

$$\langle \overline{E} \rangle_{\mathcal{I}} \overset{\text{def}}{=} \overline{\langle E \rangle_{\mathcal{I}}} \qquad \langle \bot \rangle_{\mathcal{I}} \overset{\text{def}}{=} \varnothing \qquad \langle E_1 \cup E_2 \rangle_{\mathcal{I}} \overset{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cup \langle E_2 \rangle_{\mathcal{I}}$$

$$\langle \top \rangle_{\mathcal{I}} \overset{\text{def}}{=} X \times X \qquad \langle E_1 \cap E_2 \rangle_{\mathcal{I}} \overset{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cap \langle E_2 \rangle_{\mathcal{I}}$$

Some Laws

$$R \fatsemi (S \mathbin{\bullet} T) \leq_{\mathsf{CR}} (R \fatsemi S) \mathbin{\bullet} T \qquad (R \mathbin{\bullet} S) \fatsemi T \leq_{\mathsf{CR}} R \mathbin{\bullet} (S \fatsemi T)$$

LINEAR DISTRIBUTIVITY

$$id^{\circ}_{Y} \leq_{\mathsf{CR}} \overline{R^{\dagger}} \mathbin{\bullet} R \qquad R \fatsemi \overline{R^{\dagger}} \leq_{\mathsf{CR}} id^{\bullet}_{X}$$

LINEAR ADJUNCTION

The calculus was shown to be
strictly less expressive than First Order Logic
(Lowenheim 1915)

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TARSKI'S QUESTION

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Does
a complete axiomatisation for the calculus of relations
exist?

TARSKI'S QUESTION

The calculus was shown to be
strictly less expressive than First Order Logic
(Lowenheim 1915)

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Does
a complete axiomatisation for the calculus of relations
exist?

The question has a negative answer (Monk 1961)

The Calculus of Relations IN COMPUTER SCIENCE

Rewriting (e.g. Gavazzo 2023)

Relational databases (Codd 1970)

Proof Assistants (e.g. Pous 2013) $\Longleftarrow$ Lack of variable and quantifies

Foundations of program logics (Prat 1976) (Hoare,He, 1986)

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Left Residual:

$$c ; a \leq b$$

iff

$$c \leq b ;\bullet\, \bar{a}^{\dagger}$$

$$\begin{tikzcd}
X \arrow[r, "a"] & Y \\
Z \arrow[u, "b;\bullet\bar{a}^{\dagger}"] \arrow[ur, "b"'] &
\end{tikzcd}$$

The Calculus of Relations IN COMPUTER SCIENCE

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$$\begin{tikzcd} X \arrow[r, "a"] & Y \\ Z \arrow[u, "b;\bar{a}^{\dagger}"] \arrow[ur, "b"'] & \end{tikzcd}$$

"The calculus of relations has an intrinsic charm and beauty which makes it a source of intellectual delight to all who become acquainted with it."

(Tarski 1941) (Hoare,He, 1986)

The Calculus of Neo-Peircean Relations

OVERCOMES THE MAIN LIMITATIONS OF THE CALCULUS OF RELATIONS

⟶ IT HAS THE SAME EXPRESSIVITY OF FIRST ORDER LOGIC
⟶ IT COMES WITH A COMPLETE SYSTEM OF AXIOMS

WHILE MANTAINING ITS USUAL BENEFITS:

⟶ POINT FREE REASONING: NO VARIABLES, NO QUANTIFIERS
⟶ PURELY EQUATIONAL REASONING

The Calculus of Neo-Peircean Relations

THIS IS OBTAINED BY MOVING

SYNTACTICALLY

FROM TRADITIONAL (CARTESIAN) SYNTAX

TO RESOURCE-AWARE (MONOIDAL) SYNTAX

The Calculus of Neo-Peircean Relations

THIS IS OBTAINED BY MOVING

SYNTACTICALLY

FROM TRADITIONAL (CARTESIAN) SYNTAX

TO RESOURCE-AWARE (MONOIDAL) SYNTAX

SEMANTICALLY

FROM BINARY RELATIONS $R \subseteq X \times X$

TO "MONOIDAL" RELATIONS $R \subseteq X^n \times X^m$ for some $n,m \in \mathbb{N}$

The Calculus of Neo-Peircean Relations

WITH THIS MOVE, ONE IMMEDIATELY IDENTIFIES NOVEL CONSTANTS

COPIER : FOR ALL SETS X, $\blacktriangleleft^{\circ}_{X} = \langle id_X, id_X \rangle : X \longrightarrow X \times X$

$$\blacktriangleleft^{\circ}_{X} \stackrel{\text{def}}{=} \{(x,\ (y, z)) \mid x = y \wedge x = z\}$$

The Calculus of Neo-Peircean Relations

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COPIER : FOR ALL SETS X, $\blacktriangleleft^\circ_X = \langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \longrightarrow X \times X$

$$\blacktriangleleft^\circ_X \stackrel{\text{def}}{=} \{(x, \ (y, z)) \mid x = y \wedge x = z\}$$

DISCARD : FOR ALL SETS X, $!^\circ_X : X \longrightarrow 1 \stackrel{\text{def}}{=} \{*\}$

$$!^\circ_X \stackrel{\text{def}}{=} \{(x, \star) \mid x \in X\}$$

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$$!^{\circ}_{X} \overset{\text{def}}{=} \{(x, \star) \mid x \in X\}$$

BY COMBINING THESE WITH OPPOSITE AND COMPLEMENT, ONE OBTAINS

$$\blacktriangleleft^{\circ}_{X} \overset{\text{def}}{=} \{(x,\ (y,z)) \mid x = y \wedge x = z\} \qquad !^{\circ}_{X} \overset{\text{def}}{=} \{(x, \star) \mid x \in X\}$$

$$\blacktriangleright^{\circ}_{X} \overset{\text{def}}{=} \{((y,z),\ x) \mid x = y \wedge x = z\} \qquad ¡^{\circ}_{X} \overset{\text{def}}{=} \{(\star, x) \mid x \in X\}$$

$$\blacktriangleleft^{\bullet}_{X} \overset{\text{def}}{=} \{(x,\ (y,z)) \mid x \neq y \vee x \neq z\} \qquad !^{\bullet}_{X} \overset{\text{def}}{=} \varnothing$$

$$\blacktriangleright^{\bullet}_{X} \overset{\text{def}}{=} \{((y,z),\ x) \mid x \neq y \vee x \neq z\} \qquad ¡^{\bullet}_{X} \overset{\text{def}}{=} \varnothing$$

The Calculus of Neo-Peircean Relations

WITH THIS MOVE, ONE IMMEDIATELY IDENTIFIES NOVEL OPERATIONS

MONOIDAL PRODUCTS

$$R \otimes S \stackrel{\text{def}}{=} \{(\ (x,v),(y,w)\ ) \mid (x,y) \in R \wedge (v,w) \in S\}$$

$$R \mathbin{\otimes\!\!\!\!\bullet} S \stackrel{\text{def}}{=} \{(\ (x,v),(y,w)\ ) \mid (x,y) \in R \vee (v,w) \in S\}$$

BOOLEAN OPERATIONS AS WELL AS OPPOSITE
CAN BE DERIVED BY COPIERS, DISCARDS AND PRODUCTS

The Calculus of Neo-Peircean Relations

SYNTAX

$R \in \Sigma$: A MONOIDAL SIGNATURE, NAMELY $R: n \longrightarrow m$

$$
\begin{aligned}
c ::= \; & \blacktriangleleft_1^{\circ} \mid !_1^{\circ} \mid R^{\circ} \mid ¡_1^{\circ} \mid \blacktriangleright_1^{\circ} \mid id_0^{\circ} \mid id_1^{\circ} \mid \sigma_{1,1}^{\circ} \mid c \,;\, c \mid c \otimes c \mid \\
& \blacktriangleleft_1^{\bullet} \mid !_1^{\bullet} \mid R^{\bullet} \mid ¡_1^{\bullet} \mid \blacktriangleright_1^{\bullet} \mid id_0^{\bullet} \mid id_1^{\bullet} \mid \sigma_{1,1}^{\bullet} \mid c \,;\, c \mid c \otimes c
\end{aligned}
$$

The Calculus of Neo-Peircean Relations

SYNTAX

$R \in \Sigma$: A MONOIDAL SIGNATURE, NAMELY $R: n \to m$

$$c ::= \blacktriangleleft_1^\circ \mid !_1^\circ \mid R^\circ \mid ¡_1^\circ \mid \blacktriangleright_1^\circ \mid id_0^\circ \mid id_1^\circ \mid \sigma_{1,1}^\circ \mid c \mathbin{⨾^\circ} c \mid c \otimes c \mid$$
$$\blacktriangleleft_1^\bullet \mid !_1^\bullet \mid R^\bullet \mid ¡_1^\bullet \mid \blacktriangleright_1^\bullet \mid id_0^\bullet \mid id_1^\bullet \mid \sigma_{1,1}^\bullet \mid c \mathbin{⨾^\bullet} c \mid c \mathbin{\otimes^\bullet} c$$

SEMANTICS GIVEN $\mathcal{I} = \langle X, \rho \rangle$ X IS A SET, $\rho : \Sigma_{n,m} \to \mathcal{P}(X^n \times X^m)$

$$\mathcal{I}^\sharp(\blacktriangleleft_1^\circ) \overset{\text{def}}{=} \blacktriangleleft_X^\circ \qquad \mathcal{I}^\sharp(!_1^\circ) \overset{\text{def}}{=} !_X^\circ \qquad \mathcal{I}^\sharp(\blacktriangleright_1^\circ) \overset{\text{def}}{=} \blacktriangleright_X^\circ \qquad \mathcal{I}^\sharp(¡_1^\circ) \overset{\text{def}}{=} ¡_X^\circ$$
$$\mathcal{I}^\sharp(id_0^\circ) \overset{\text{def}}{=} id_{\mathbb{1}}^\circ \qquad \mathcal{I}^\sharp(id_1^\circ) \overset{\text{def}}{=} id_X^\circ \qquad \mathcal{I}^\sharp(\sigma_{1,1}^\circ) \overset{\text{def}}{=} \sigma_{X,X}^\circ \qquad \mathcal{I}^\sharp(R^\circ) \overset{\text{def}}{=} \rho(R)$$
$$\mathcal{I}^\sharp(c \mathbin{⨾^\circ} d) \overset{\text{def}}{=} \mathcal{I}^\sharp(c) \mathbin{⨾^\circ} \mathcal{I}^\sharp(d) \qquad \mathcal{I}^\sharp(c \otimes d) \overset{\text{def}}{=} \mathcal{I}^\sharp(c) \otimes \mathcal{I}^\sharp(d) \qquad \mathcal{I}^\sharp(R^\bullet) \overset{\text{def}}{=} \overline{\rho(R)}^\dagger$$

● STANDS FOR EITHER ∘ OR ●

The Calculus of Neo-Peircean Relations

COMPLETE AXIOMATISATION

Axioms of strict symmetric monoidal categories

$$a \fatsemi (b \fatsemi c) = (a \fatsemi b) \fatsemi c \qquad id^{\circ}_n \fatsemi c = c = c \fatsemi id^{\circ}_m \qquad (a \circledast b) \circledast c = a \circledast (b \circledast c) \qquad id^{\circ}_0 \circledast c = c = id^{\circ}_0 \circledast c$$

$$(a \circledast b) \fatsemi (c \circledast d) = (a \fatsemi c) \circledast (b \fatsemi d) \qquad \sigma^{\circ}_{1,1} \fatsemi \sigma^{\circ}_{1,1} = id^{\circ}_2 \qquad (c \circledast id^{\circ}_o) \fatsemi \sigma^{\circ}_{m,o} = \sigma^{\circ}_{n,o} \fatsemi (id^{\circ}_o \circledast c)$$

Axioms of cartesian bicategories

$$\blacktriangleleft^{\circ}_n \fatsemi (id^{\circ}_n \otimes \blacktriangleleft^{\circ}_n) \overset{(\blacktriangleleft^{\circ}\text{-as})}{=} \blacktriangleleft^{\circ}_n \fatsemi (\blacktriangleleft^{\circ}_n \otimes id^{\circ}_n) \qquad (id^{\circ}_n \otimes \blacktriangleright^{\circ}_n) \fatsemi \blacktriangleright^{\circ}_n \overset{(\blacktriangleright^{\circ}\text{-as})}{=} (\blacktriangleright^{\circ}_n \otimes id^{\circ}_n) \fatsemi \blacktriangleright^{\circ}_n$$

$$\blacktriangleleft^{\circ}_n \fatsemi (id^{\circ}_n \otimes !^{\circ}_n) \overset{(\blacktriangleleft^{\circ}\text{-un})}{=} id^{\circ}_n \qquad (id^{\circ}_n \otimes \text{¡}^{\circ}_n) \fatsemi \blacktriangleright^{\circ}_n \overset{(\blacktriangleright^{\circ}\text{-un})}{=} id^{\circ}_n$$

$$\blacktriangleleft^{\circ}_n \fatsemi \sigma^{\circ}_{n,n} \overset{(\blacktriangleleft^{\circ}\text{-co})}{=} \blacktriangleleft^{\circ}_n \qquad \sigma^{\circ}_{n,n} \fatsemi \blacktriangleright^{\circ}_n \overset{(\blacktriangleright^{\circ}\text{-co})}{=} \blacktriangleright^{\circ}_n$$

$$(\blacktriangleleft^{\circ}_n \otimes id^{\circ}_n) \fatsemi (id^{\circ}_n \otimes \blacktriangleright^{\circ}_n) \overset{(\mathrm{F}^{\circ})}{=} (id^{\circ}_n \otimes \blacktriangleleft^{\circ}_n) \fatsemi (\blacktriangleright^{\circ}_n \otimes id^{\circ}_n) \qquad \blacktriangleleft^{\circ}_n \fatsemi \blacktriangleright^{\circ}_n \overset{(\mathrm{S}^{\circ})}{=} id^{\circ}_n$$

$$\text{¡}^{\circ}_n \fatsemi !^{\circ}_n \overset{(\epsilon !^{\circ})}{\leq} id^{\circ}_0 \qquad \blacktriangleright^{\circ}_n \fatsemi \blacktriangleleft^{\circ}_n \overset{(\epsilon \blacktriangleleft^{\circ})}{\leq} (id^{\circ}_n \otimes id^{\circ}_n) \qquad c \fatsemi \blacktriangleleft^{\circ}_m \overset{(\blacktriangleleft^{\circ}\text{-nat})}{\leq} \blacktriangleleft^{\circ}_n \fatsemi (c \otimes c)$$

$$id^{\circ}_n \overset{(\eta !^{\circ})}{\leq} !^{\circ}_n \fatsemi \text{¡}^{\circ}_n \qquad id^{\circ}_n \overset{(\eta \blacktriangleleft^{\circ})}{\leq} \blacktriangleleft^{\circ}_n \fatsemi \blacktriangleright^{\circ}_n \qquad c \fatsemi !^{\circ}_m \overset{(!^{\circ}\text{-nat})}{\leq} !^{\circ}_n$$

Axioms of cocartesian bicategories

$$\blacktriangleleft^{\bullet}_n \mathbin{\boldsymbol{;}} (id^{\bullet}_n \circledast \blacktriangleleft^{\bullet}_n) \overset{(\blacktriangleleft^{\bullet}\text{-as})}{=} \blacktriangleleft^{\bullet}_n \mathbin{\boldsymbol{;}} (\blacktriangleleft^{\bullet}_n \circledast id^{\bullet}_n) \qquad (id^{\bullet}_n \circledast \blacktriangleright^{\bullet}_n) \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_n \overset{(\blacktriangleright^{\bullet}\text{-as})}{=} (\blacktriangleright^{\bullet}_n \circledast id^{\bullet}_n) \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_n$$

$$\blacktriangleleft^{\bullet}_n \mathbin{\boldsymbol{;}} (id^{\bullet}_n \circledast !^{\bullet}_n) \overset{(\blacktriangleleft^{\bullet}\text{-un})}{=} id^{\bullet}_n \qquad (id^{\bullet}_n \circledast \text{¡}^{\bullet}_n) \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_n \overset{(\blacktriangleright^{\bullet}\text{-un})}{=} id^{\bullet}_n$$

$$\blacktriangleleft^{\bullet}_n \mathbin{\boldsymbol{;}} \sigma^{\bullet}_{n,n} \overset{(\blacktriangleleft^{\bullet}\text{-co})}{=} \blacktriangleleft^{\bullet}_n \qquad \sigma^{\bullet}_{n,n} \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_n \overset{(\blacktriangleright^{\bullet}\text{-co})}{=} \blacktriangleright^{\bullet}_n$$

$$(\blacktriangleleft^{\bullet}_n \circledast id^{\bullet}_n) \mathbin{\boldsymbol{;}} (id^{\bullet}_n \circledast \blacktriangleright^{\bullet}_n) \overset{(\mathrm{F}^{\bullet})}{=} (id^{\bullet}_n \circledast \blacktriangleleft^{\bullet}_n) \mathbin{\boldsymbol{;}} (\blacktriangleright^{\bullet}_n \circledast id^{\bullet}_n) \qquad \blacktriangleleft^{\bullet}_n \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_n \overset{(\mathrm{S}^{\bullet})}{=} id^{\bullet}_n$$

$$!^{\bullet}_n \mathbin{\boldsymbol{;}} \text{¡}^{\bullet}_n \overset{(\epsilon \text{¡}^{\bullet})}{\leq} id^{\bullet}_n \qquad \blacktriangleleft^{\bullet}_n \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_n \overset{(\epsilon \blacktriangleright^{\bullet})}{\leq} id^{\bullet}_n \qquad \blacktriangleleft^{\bullet}_n \mathbin{\boldsymbol{;}} (c \circledast c) \overset{(\blacktriangleleft^{\bullet}\text{-nat})}{\leq} c \mathbin{\boldsymbol{;}} \blacktriangleleft^{\bullet}_m$$

$$id^{\bullet}_0 \overset{(\eta \text{¡}^{\bullet})}{\leq} \text{¡}^{\bullet}_n \mathbin{\boldsymbol{;}} !^{\bullet}_n \qquad \blacktriangleright^{\bullet}_n \mathbin{\boldsymbol{;}} \blacktriangleleft^{\bullet}_n \overset{(\eta \blacktriangleright^{\bullet})}{\leq} (id^{\bullet}_n \circledast id^{\bullet}_n) \qquad !^{\bullet}_n \overset{(!^{\bullet}\text{-nat})}{\leq} c \mathbin{\boldsymbol{;}} !^{\bullet}_m$$

Axioms of closed symmetric monoidal linear bicategories

$$a \fatsemi (b \mathbin{\boldsymbol{;}} c) \overset{(\delta_l)}{\leq} (a \fatsemi b) \mathbin{\boldsymbol{;}} c \qquad (a \mathbin{\boldsymbol{;}} b) \fatsemi c \overset{(\delta_r)}{\leq} a \mathbin{\boldsymbol{;}} (b \fatsemi c)$$

$$id^{\circ}_{n+m} \overset{(\tau\sigma^{\circ})}{\leq} \sigma^{\circ}_{n,m} \mathbin{\boldsymbol{;}} \sigma^{\bullet}_{m,n} \qquad \sigma^{\bullet}_{n,m} \fatsemi \sigma^{\circ}_{m,n} \overset{(\gamma\sigma^{\circ})}{\leq} id^{\bullet}_{n+m} \qquad id^{\circ}_n \overset{(\tau R^{\circ})}{\leq} R^{\circ} \mathbin{\boldsymbol{;}} R^{\bullet} \qquad R^{\bullet} \fatsemi R^{\circ} \overset{(\gamma R^{\circ})}{\leq} id^{\bullet}_m$$

$$id^{\circ}_{n+m} \overset{(\tau\sigma^{\bullet})}{\leq} \sigma^{\bullet}_{n,m} \mathbin{\boldsymbol{;}} \sigma^{\circ}_{m,n} \qquad \sigma^{\circ}_{n,m} \fatsemi \sigma^{\bullet}_{m,n} \overset{(\gamma\sigma^{\bullet})}{\leq} id^{\bullet}_{n+m} \qquad id^{\circ}_m \overset{(\tau R^{\bullet})}{\leq} R^{\bullet} \mathbin{\boldsymbol{;}} R^{\circ} \qquad R^{\circ} \fatsemi R^{\bullet} \overset{(\gamma R^{\bullet})}{\leq} id^{\bullet}_n$$

$$id^{\circ}_{n+m} \overset{(\circledast^{\circ})}{\leq} id^{\circ}_n \circledast id^{\circ}_m \qquad id^{\bullet}_n \otimes id^{\bullet}_m \overset{(\otimes^{\bullet})}{\leq} id^{\bullet}_{n+m}$$

$$(a \mathbin{\boldsymbol{;}} b) \otimes (c \mathbin{\boldsymbol{;}} d) \overset{(\nu^{\circ}_l)}{\leq} (a \otimes c) \mathbin{\boldsymbol{;}} (b \circledast d) \qquad (a \circledast c) \fatsemi (b \otimes d) \overset{(\nu^{\bullet}_l)}{\leq} (a \fatsemi b) \circledast (c \fatsemi d)$$

$$(a \mathbin{\boldsymbol{;}} b) \otimes (c \mathbin{\boldsymbol{;}} d) \overset{(\nu^{\circ}_r)}{\leq} (a \circledast c) \mathbin{\boldsymbol{;}} (b \otimes d) \qquad (a \otimes c) \fatsemi (b \circledast d) \overset{(\nu^{\bullet}_r)}{\leq} (a \fatsemi b) \circledast (c \fatsemi d)$$

Additional axioms for fo-bicategories

$$id^{\circ}_n \overset{(\tau \blacktriangleleft^{\circ})}{\leq} \blacktriangleleft^{\circ}_n \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_n \qquad \blacktriangleright^{\bullet}_n \fatsemi \blacktriangleleft^{\circ}_n \overset{(\gamma \blacktriangleleft^{\circ})}{\leq} id^{\bullet}_{n+n} \qquad id^{\circ}_{n+n} \overset{(\tau \blacktriangleright^{\circ})}{\leq} \blacktriangleright^{\circ}_n \mathbin{\boldsymbol{;}} \blacktriangleleft^{\bullet}_n \qquad \blacktriangleleft^{\bullet}_n \fatsemi \blacktriangleright^{\circ}_n \overset{(\gamma \blacktriangleright^{\circ})}{\leq} id^{\bullet}_n$$

$$id^{\circ}_n \overset{(\tau !^{\circ})}{\leq} !^{\circ}_n \mathbin{\boldsymbol{;}} \text{¡}^{\bullet}_n \qquad \text{¡}^{\bullet}_n \fatsemi !^{\circ}_n \overset{(\gamma !^{\circ})}{\leq} id^{\bullet}_0 \qquad id^{\circ}_0 \overset{(\tau \text{¡}^{\circ})}{\leq} \text{¡}^{\circ}_n \mathbin{\boldsymbol{;}} !^{\bullet}_n \qquad !^{\bullet}_n \fatsemi \text{¡}^{\circ}_n \overset{(\gamma \text{¡}^{\circ})}{\leq} id^{\bullet}_n$$

$$id^{\circ}_n \overset{(\tau \blacktriangleleft^{\bullet})}{\leq} \blacktriangleleft^{\bullet}_n \mathbin{\boldsymbol{;}} \blacktriangleright^{\circ}_n \qquad \blacktriangleright^{\circ}_n \fatsemi \blacktriangleleft^{\bullet}_n \overset{(\gamma \blacktriangleleft^{\bullet})}{\leq} id^{\bullet}_{n+n} \qquad id^{\circ}_{n+n} \overset{(\tau \blacktriangleright^{\bullet})}{\leq} \blacktriangleright^{\bullet}_n \mathbin{\boldsymbol{;}} \blacktriangleleft^{\circ}_n \qquad \blacktriangleleft^{\circ}_n \fatsemi \blacktriangleright^{\bullet}_n \overset{(\gamma \blacktriangleright^{\bullet})}{\leq} id^{\bullet}_n$$

$$id^{\circ}_n \overset{(\tau !^{\bullet})}{\leq} !^{\bullet}_n \mathbin{\boldsymbol{;}} \text{¡}^{\circ}_n \qquad \text{¡}^{\circ}_n \fatsemi !^{\bullet}_n \overset{(\gamma !^{\bullet})}{\leq} id^{\bullet}_0 \qquad id^{\circ}_0 \overset{(\tau \text{¡}^{\bullet})}{\leq} \text{¡}^{\bullet}_n \mathbin{\boldsymbol{;}} !^{\circ}_n \qquad !^{\circ}_n \fatsemi \text{¡}^{\bullet}_n \overset{(\gamma \text{¡}^{\bullet})}{\leq} id^{\bullet}_0$$

$$(\blacktriangleleft^{\bullet}_n \otimes id^{\circ}_n) \fatsemi (id^{\circ}_n \otimes \blacktriangleright^{\circ}_n) \overset{(\mathrm{F}^{\bullet}{}_{\circ})}{=} (id^{\circ}_n \otimes \blacktriangleleft^{\circ}_n) \fatsemi (\blacktriangleright^{\bullet}_n \otimes id^{\circ}_n) \qquad (\blacktriangleleft^{\circ}_n \otimes id^{\circ}_n) \fatsemi (id^{\circ}_n \otimes \blacktriangleright^{\bullet}_n) \overset{(\mathrm{F}^{\circ}{}_{\bullet})}{=} (id^{\circ}_n \otimes \blacktriangleleft^{\bullet}_n) \fatsemi (\blacktriangleright^{\circ}_n \otimes id^{\circ}_n)$$

$$(\blacktriangleleft^{\circ}_n \circledast id^{\bullet}_n) \mathbin{\boldsymbol{;}} (id^{\bullet}_n \circledast \blacktriangleright^{\bullet}_n) \overset{(\mathrm{F}_{\bullet}{}^{\circ})}{=} (id^{\bullet}_n \circledast \blacktriangleleft^{\bullet}_n) \mathbin{\boldsymbol{;}} (\blacktriangleright^{\circ}_n \circledast id^{\bullet}_n) \qquad (\blacktriangleleft^{\bullet}_n \circledast id^{\bullet}_n) \mathbin{\boldsymbol{;}} (id^{\bullet}_n \circledast \blacktriangleright^{\circ}_n) \overset{(\mathrm{F}_{\circ}{}^{\bullet})}{=} (id^{\bullet}_n \circledast \blacktriangleleft^{\circ}_n) \mathbin{\boldsymbol{;}} (\blacktriangleright^{\bullet}_n \circledast id^{\bullet}_n)$$

MAIN THEOREM

Let E, F be two expressions.

$E \boxed{\leq} F$ iff $\forall \mathcal{I},\ \mathcal{I}^{\#}(E) \leq \mathcal{I}^{\#}(F)$

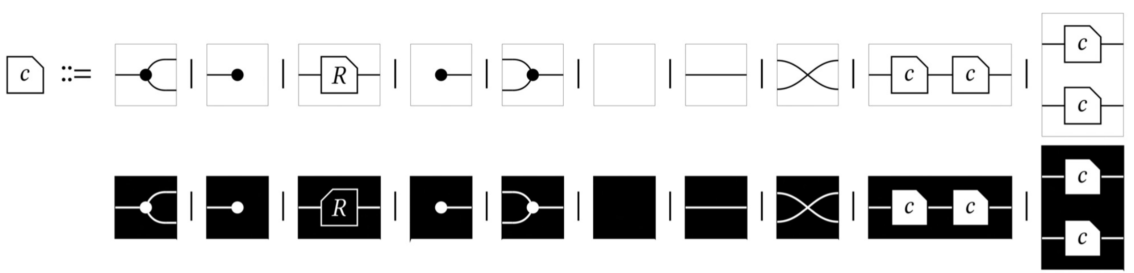
GENERATED BY

Diagrammatic Syntax

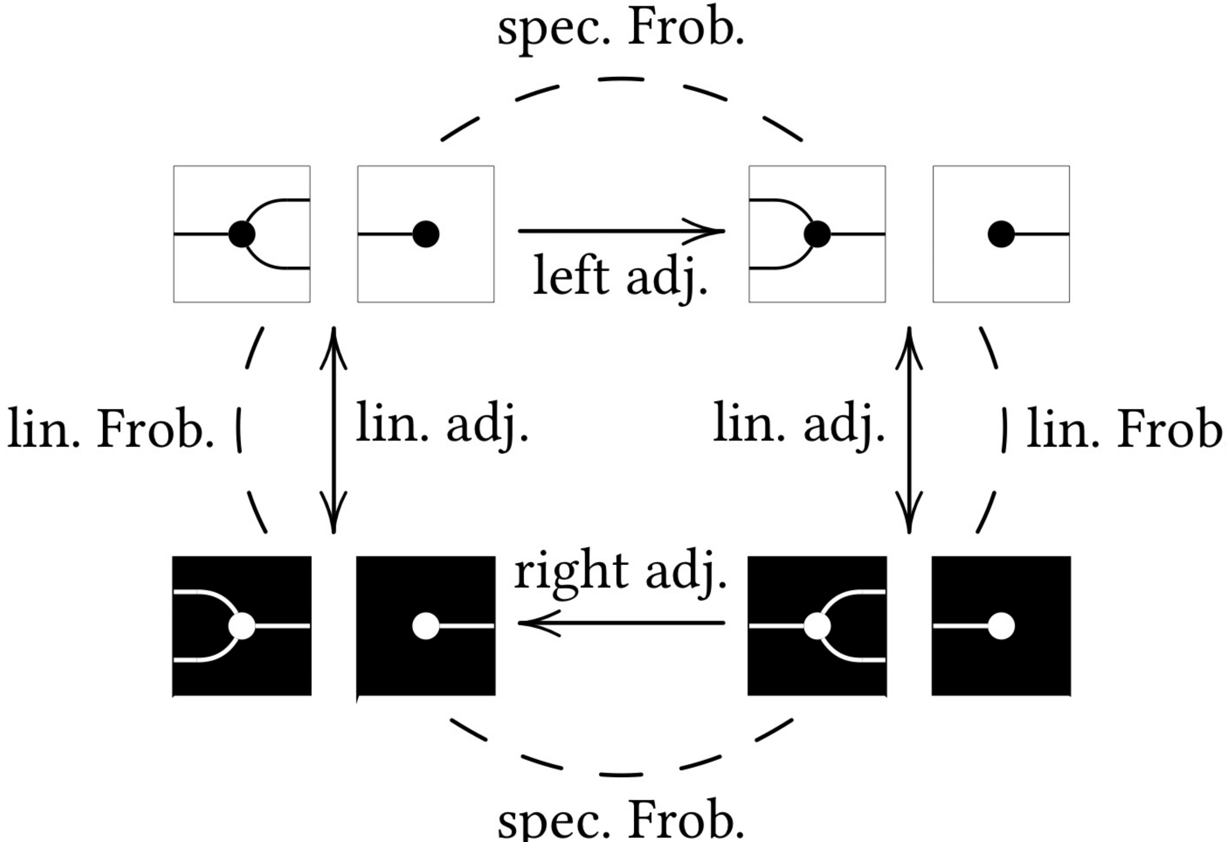
TERMS

$$
\begin{aligned}
c ::= {} & \blacktriangleleft^{\circ}_{1} \mid {!}^{\circ}_{1} \mid R^{\circ} \mid {¡}^{\circ}_{1} \mid \blacktriangleright^{\circ}_{1} \mid id^{\circ}_{0} \mid id^{\circ}_{1} \mid \sigma^{\circ}_{1,1} \mid c \overset{\circ}{;} c \mid c \otimes c \mid \\
& \blacktriangleleft^{\bullet}_{1} \mid {!}^{\bullet}_{1} \mid R^{\bullet} \mid {¡}^{\bullet}_{1} \mid \blacktriangleright^{\bullet}_{1} \mid id^{\bullet}_{0} \mid id^{\bullet}_{1} \mid \sigma^{\bullet}_{1,1} \mid c \overset{\bullet}{;} c \mid c \circledast c
\end{aligned}
$$

DIAGRAMS



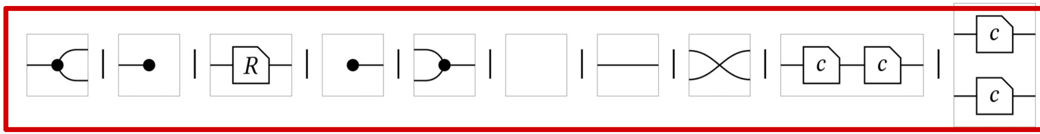
THE TAO OF LOGIC



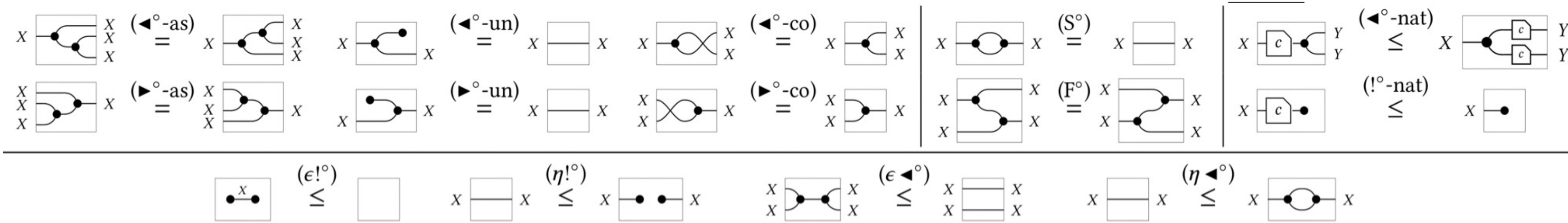
CARTESIAN BICATEGORIES

(CARBONI, WALTERS 1987)

Operations



Laws

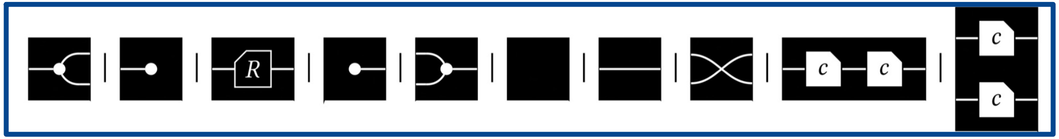


MAIN EXAMPLE: Rel°, sets and relations with the white structure

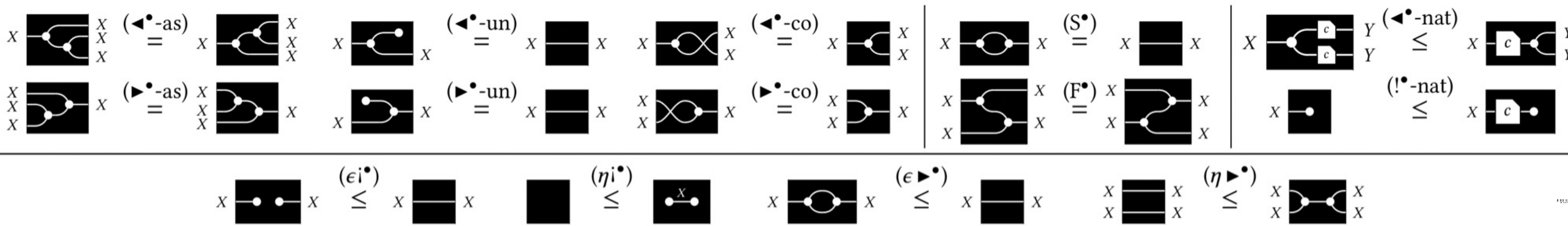
INTERNAL LANGUAGE: The regular fragment of FOL $(\exists, \wedge, \top, =)$

COCARTESIAN BICATEGORIES

Operations



Laws



MAIN EXAMPLE: $\mathsf{Rel}^\bullet$, sets and relations
with the black structure

INTERNAL LANGUAGE: The coregular fragment of FOL
$(\forall, \vee, F, \neq)$

CARTESIAN BICATEGORIES

↕

REGULAR LOGIC

COCARTESIAN BICATEGORIES

↕

COREGULAR LOGIC

CARTESIAN BICATEGORIES
$\Updownarrow$
REGULAR LOGIC

COCARTESIAN BICATEGORIES
$\Updownarrow$
COREGULAR LOGIC

To make them interact one needs some categorical structures dealing with two compositions ($\fatsemi$, $\bullet\!\!\!\!;$)

LINEAR BICATEGORIES (Cockett, Seely, 1997)

CARTESIAN BICATEGORIES
$\updownarrow$
REGULAR LOGIC

$\longleftrightarrow$

COCARTESIAN BICATEGORIES
$\updownarrow$
COREGULAR LOGIC

To make them interact one needs some categorical structures dealing with two compositions $(; \, , \, \boldsymbol{;})$

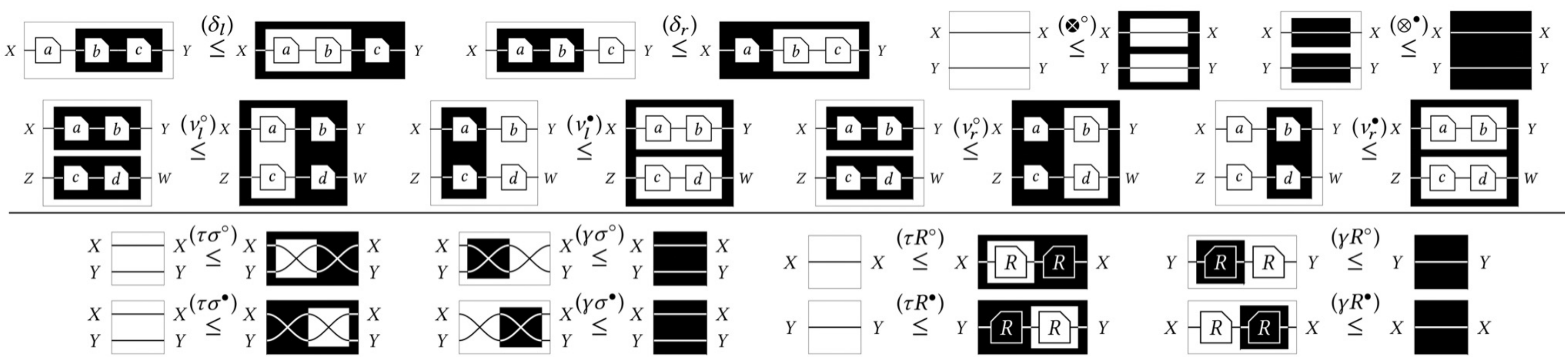
LINEAR BICATEGORIES

(Cockett, Seely, 1997)

CARTESIAN BICATEGORIES
↕
REGULAR LOGIC

COCARTESIAN BICATEGORIES
↕
COREGULAR LOGIC

To make them interact one needs some categorical structures dealing with two compositions $(\,;^{\circ}\,,\,;^{\bullet}\,)$



LINEAR BICATEGORIES

(Cockett, Seely, 1997)

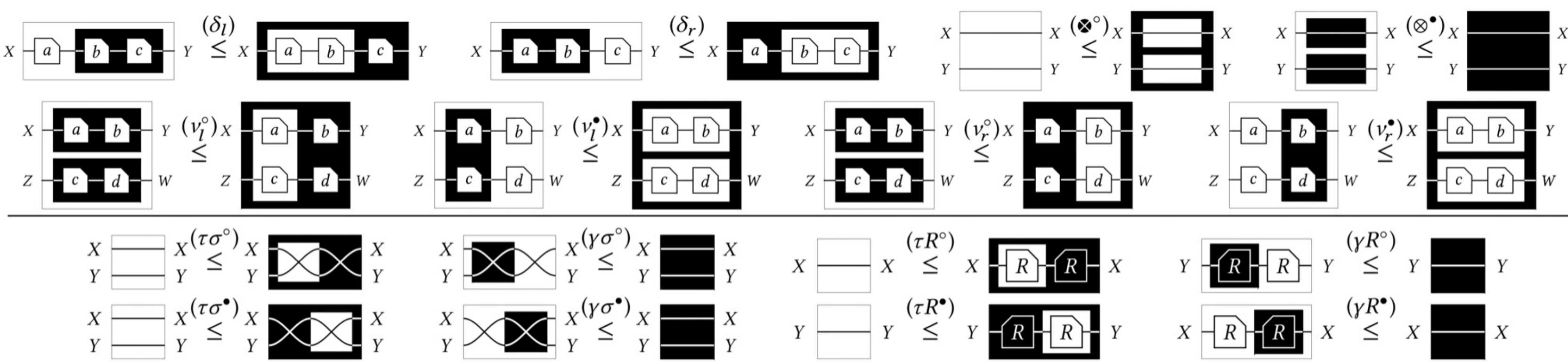
CARTESIAN BICATEGORIES ↕ **REGULAR LOGIC**

↔

COCARTESIAN BICATEGORIES ↕ **COREGULAR LOGIC**

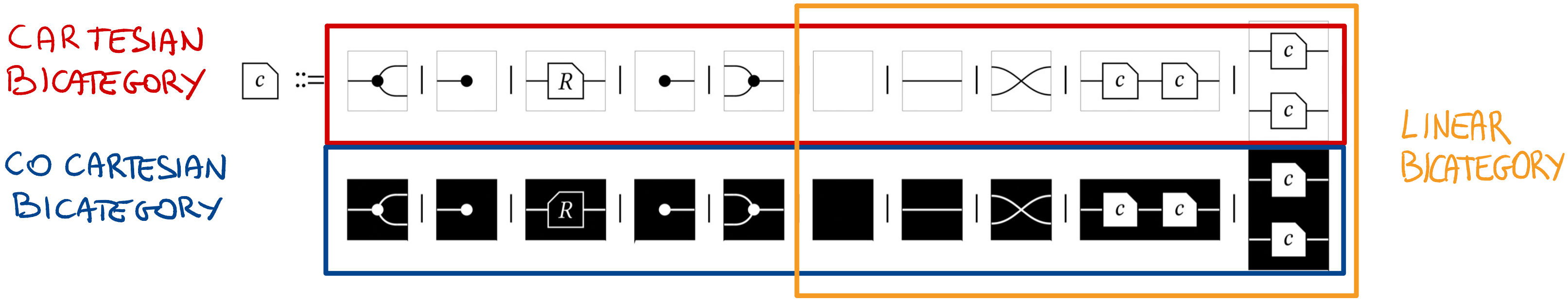
To make them interact one needs some categorical structure dealing with two compositions (; , ;)

Dialectica, Linear distributive (De Paiva 1991, Cockett-Seely 1997)



FIRST ORDER BICATEGORIES

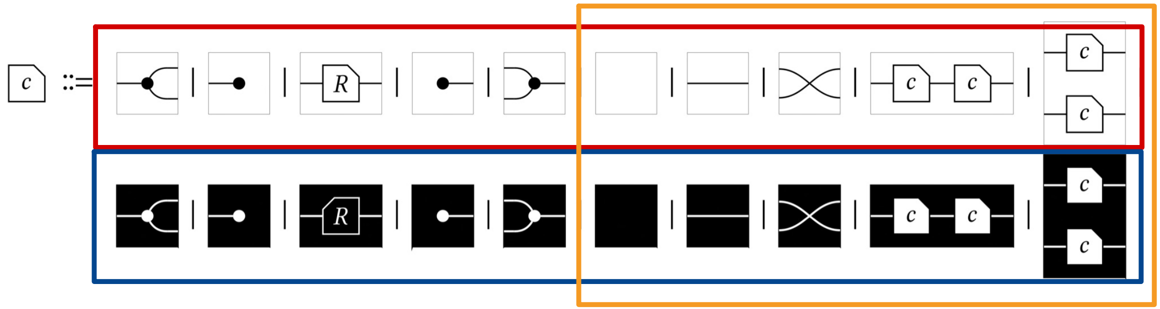
main example Rel



FIRST ORDER BICATEGORIES main example Rel

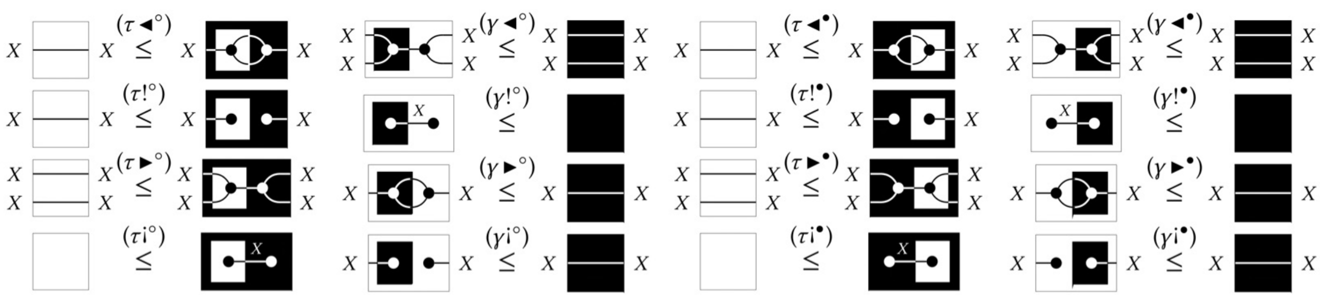
CARTESIAN BICATEGORY

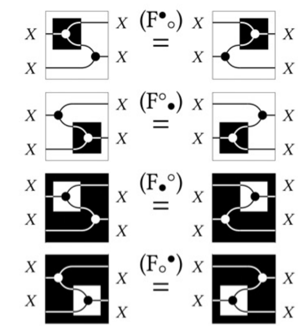
CO CARTESIAN BICATEGORY



LINEAR BICATEGORY

SUCH THAT





WHITE COMONOID IS LINEAR ADJOINT TO BLACK MONOID

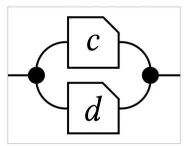
BLACK COMONOID IS LINEAR ADJOINT TO WHITE MONOID

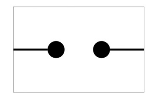
LINEAR FROBENIUS

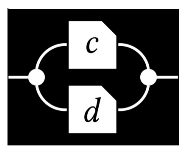
FIRST ORDER BICATEGORIES: PROPERTIES

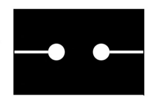
PROPOSITION A

In a first-order bicategory $\mathcal{C}$, each homset $\mathcal{C}[X, Y]$ carries a boolean algebra

$c \sqcap d \stackrel{\text{def}}{=}$ X  Y

$\top \stackrel{\text{def}}{=}$ X  Y

$c \sqcup d \stackrel{\text{def}}{=}$ X  Y

$\bot \stackrel{\text{def}}{=}$ X  Y

FIRST ORDER BICATEGORIES: PROPERTIES

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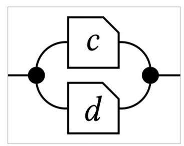
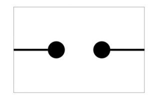
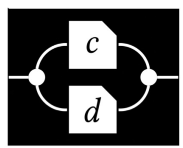
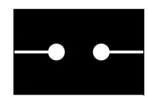
$c \sqcap d \stackrel{\text{def}}{=}$ X  Y

$\top \stackrel{\text{def}}{=}$ X  Y

$c \sqcup d \stackrel{\text{def}}{=}$ X  Y

$\bot \stackrel{\text{def}}{=}$ X  Y

$\overline{(\cdot)} : (\mathcal{C}, ;^{\circ}, ;^{\bullet}) \xrightarrow{\cong} (\mathcal{C}^{co}, ;^{\bullet}, ;^{\circ})$ swaps black and white

FIRST ORDER BICATEGORIES: PROPERTIES

PROPOSITION A

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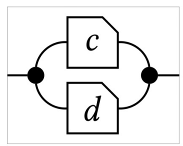
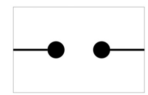
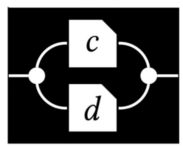
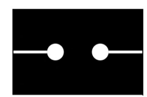
$c \sqcap d \overset{\text{def}}{=} X$  Y $\qquad \top \overset{\text{def}}{=} X$  Y

$c \sqcup d \overset{\text{def}}{=} X$  Y $\qquad \bot \overset{\text{def}}{=} X$  Y

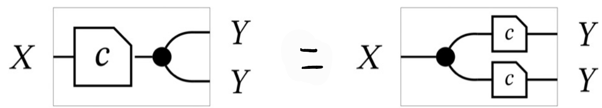
$\overline{(\cdot)} : (\mathcal{C}, \overset{\circ}{;}, \overset{\bullet}{;}) \xrightarrow{\cong} (\mathcal{C}^{co}, \overset{\bullet}{;}, \overset{\circ}{;})$ swaps black and white

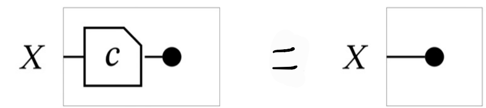
$(\cdot)^{\dagger} : (\mathcal{C}, \overset{\circ}{;}, \overset{\bullet}{;}) \xrightarrow{\cong} (\mathcal{C}^{op}, \overset{\circ}{;}, \overset{\bullet}{;})$ swaps left and right

FIRST ORDER BICATEGORIES: PROPERTIES

DEFINITION

An arrow $c : X \to Y$ is said to be a map if:



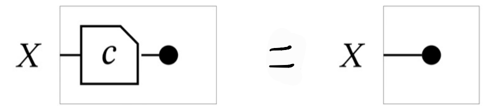


EXAMPLE In Rel maps coincide with functions

FIRST ORDER BICATEGORIES: PROPERTIES

DEFINITION

An arrow $c : X \to Y$ is said to be a map if:

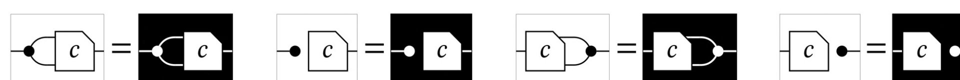


EXAMPLE In Rel maps coincide with functions

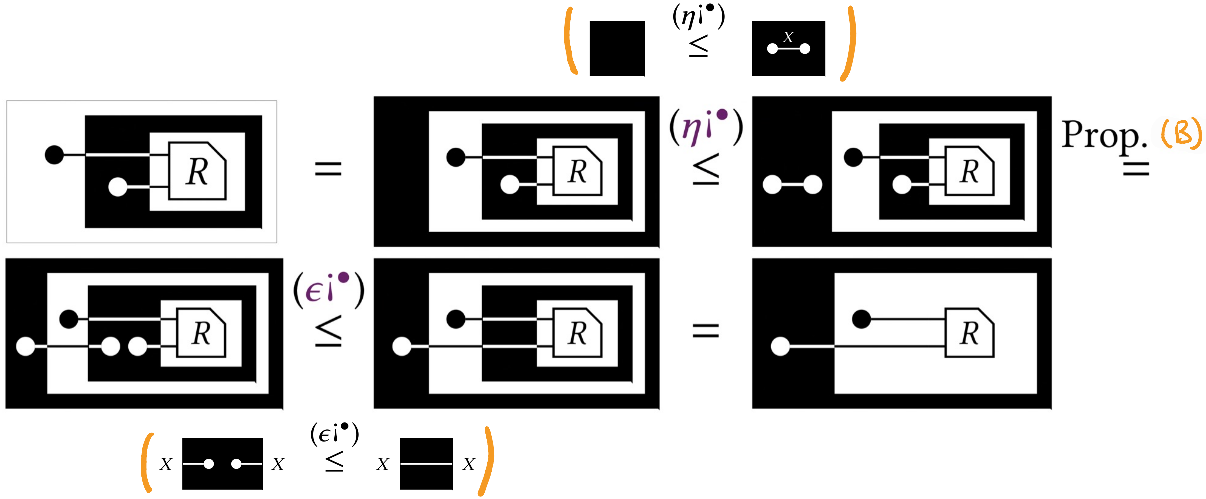
PROPOSITION B

For all maps $f : X \to Y$ and arrows $c : Y \to Z$, $f \,\mathring{;}\, c = \bar{f} \,;\, c$

and thus



Proofs as diagram rewrites



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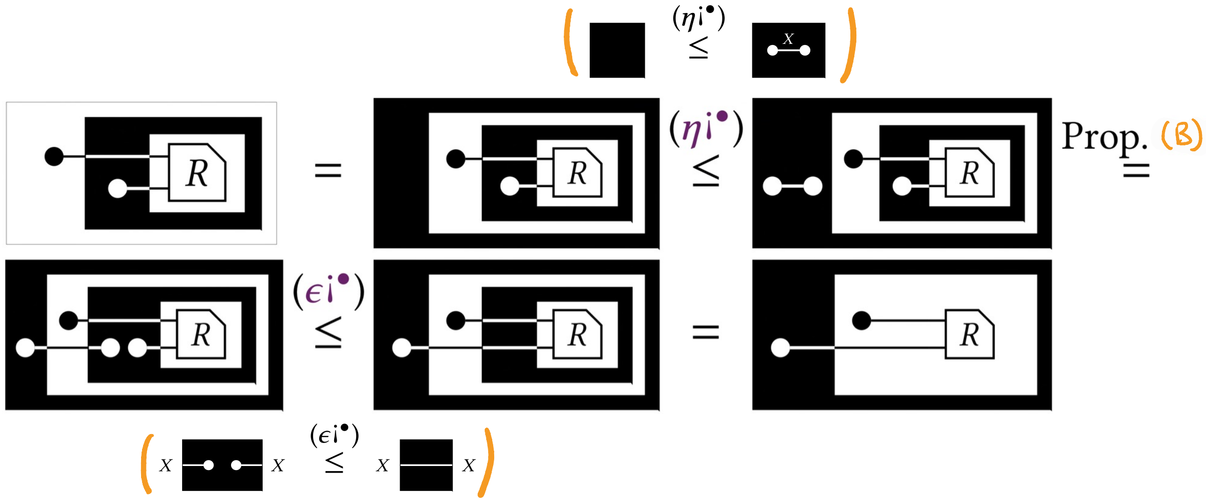
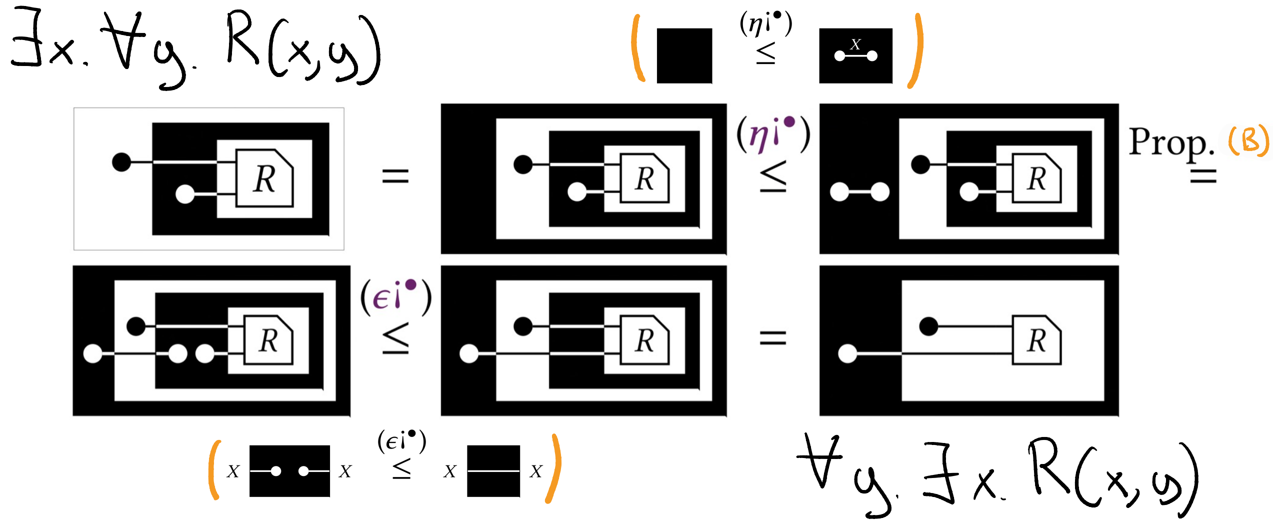


(IN THE PROPOSITIONAL CASE, THE AXIOMS OF FO-BICATEGORIES GIVE RISE TO THE DEEP INFERENCE SYSTEM SKSg (Brunneler 2003))

Proofs as diagram rewrites



(IN THE PROPOSITIONAL CASE, THE AXIOMS OF FO-BICATEGORIES GIVE

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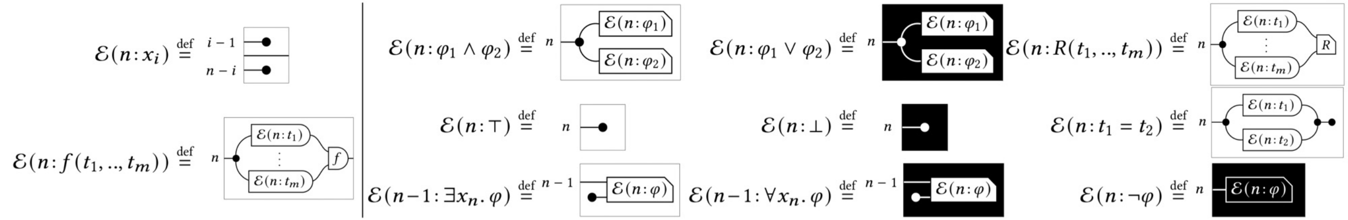
Compositional encodings

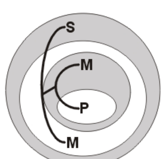


FIRST-ORDER LOGIC

CALCULUS OF RELATIONS

CALCULUS OF NEO-PEIRCEAN RELATIONS

PEIRCE'S EXISTENTIAL GRAPHS (1909)



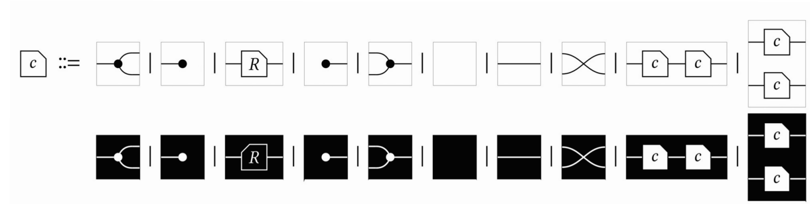
QUINE'S PREDICATE-FUNCTOR LOGIC (1971)

Functorial Semantics

Given a monoidal signature Σ, one can freely generate a FO-bicategory FOB_Σ

OBJECTS : are natural numbers

ARROWS : are diagrams
modulo the axioms



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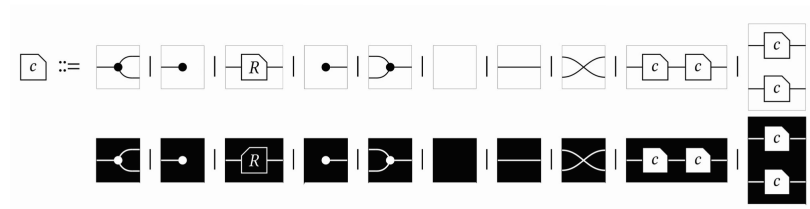

A morphism of FO-bicategories $\mathcal{M} : FOB_\Sigma \longrightarrow Rel$ gives rise to an interpretation (X, ρ) where

$$X = \mathcal{M}(1)$$

$$\rho(R) = \mathcal{M}(R) \qquad \text{for all } R \in \Sigma$$

Functorial Semantics

An interpretation $\mathcal{I} = (X, \rho)$, give rise to a morphism of FO-bicategory $\mathcal{I}^\sharp : FOB_\Sigma \longrightarrow Rel$ defined as the semantics of NPR_Σ

$$
\begin{array}{llll}
\mathcal{I}^\sharp(\blacktriangleleft_1^\circ) \stackrel{\text{def}}{=} \blacktriangleleft_X^\circ & \mathcal{I}^\sharp(!_1^\circ) \stackrel{\text{def}}{=} !_X^\circ & \mathcal{I}^\sharp(\blacktriangleright_1^\circ) \stackrel{\text{def}}{=} \blacktriangleright_X^\circ & \mathcal{I}^\sharp(¡_1^\circ) \stackrel{\text{def}}{=} ¡_X^\circ \\
\mathcal{I}^\sharp(id_0^\circ) \stackrel{\text{def}}{=} id_{\mathbb{1}}^\circ & \mathcal{I}^\sharp(id_1^\circ) \stackrel{\text{def}}{=} id_X^\circ & \mathcal{I}^\sharp(\sigma_{1,1}^\circ) \stackrel{\text{def}}{=} \sigma_{X,X}^\circ & \mathcal{I}^\sharp(R^\circ) \stackrel{\text{def}}{=} \rho(R) \\
\mathcal{I}^\sharp(c ; d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) ; \mathcal{I}^\sharp(d) & \mathcal{I}^\sharp(c \otimes d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) \otimes \mathcal{I}^\sharp(d) & & \mathcal{I}^\sharp(R^\bullet) \stackrel{\text{def}}{=} \overline{\rho(R)}^\dagger
\end{array}
$$

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$$
\begin{array}{llll}
\mathcal{I}^\sharp(\blacktriangleleft^\circ_1) \stackrel{\text{def}}{=} \blacktriangleleft^\circ_X & \mathcal{I}^\sharp(!^\circ_1) \stackrel{\text{def}}{=} !^\circ_X & \mathcal{I}^\sharp(\blacktriangleright^\circ_1) \stackrel{\text{def}}{=} \blacktriangleright^\circ_X & \mathcal{I}^\sharp(¡^\circ_1) \stackrel{\text{def}}{=} ¡^\circ_X \\
\mathcal{I}^\sharp(id^\circ_0) \stackrel{\text{def}}{=} id^\circ_{\mathbb{1}} & \mathcal{I}^\sharp(id^\circ_1) \stackrel{\text{def}}{=} id^\circ_X & \mathcal{I}^\sharp(\sigma^\circ_{1,1}) \stackrel{\text{def}}{=} \sigma^\circ_{X,X} & \mathcal{I}^\sharp(R^\circ) \stackrel{\text{def}}{=} \rho(R) \\
\mathcal{I}^\sharp(c ; d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) ; \mathcal{I}^\sharp(d) & \mathcal{I}^\sharp(c \otimes d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) \otimes \mathcal{I}^\sharp(d) & & \mathcal{I}^\sharp(R^\bullet) \stackrel{\text{def}}{=} \overline{\rho(R)}^\dagger
\end{array}
$$

BIJECTION

A morphism of FO-bicategories $\mathcal{M} : \mathsf{FOB}_\Sigma \to \mathsf{Rel}$ gives rise to an interpretation (X, ρ) where

$$X = \mathcal{M}(1)$$

$$\rho(R) = \mathcal{M}(R) \quad \text{for all } R \in \Sigma$$

Functorial Semantics

Similarly, given a "first order theory" (Σ, E), one can freely generate a FO-bicategory $FOB_{\Sigma,E}$

OBJECTS: are natural numbers

ARROWS: are diagrams modulo the axioms and the equations in E

Functorial Semantics

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morphisms $\mathcal{M} : FOB_{\Sigma,E} \longrightarrow Rel$

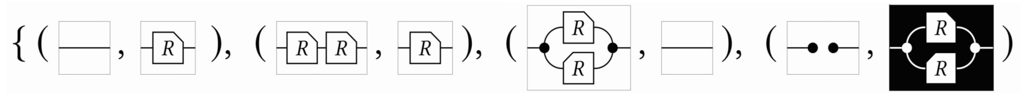
ARE IN BIJECTIVE CORRESPONDENCE WITH

models of (Σ, E)

Functorial Semantics

EXAMPLE

$\Sigma = \{ R : 1 \to 1 \}$

$E =$ 

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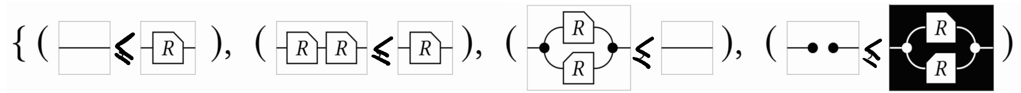
$E =$ 

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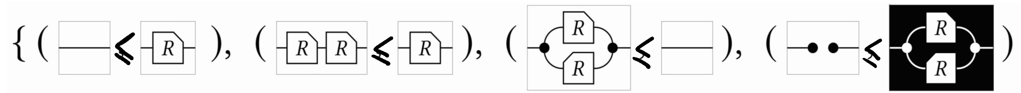
$\Sigma = \{ R : 1 \to 1 \}$ $\quad E =$ 

An interpretation of Σ consists of a set X and a relation $R \subseteq X \times X$

A model of (Σ, E) is a "total" order

---

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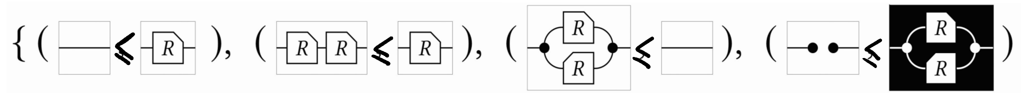
Functorial Semantics

EXAMPLE

$\Sigma = \{ R : 1 \to 1 \}$

REFLEXIVITY TRANSITIVITY ANTISYMMETRY TOTALITY

$E =$ 

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A FOX THEOREM OF CLASSICAL LOGIC

$$\begin{tikzcd} \mathbb{CB} \arrow[r, bend left, "\mathrm{HmI}"] \arrow[r, phantom, "\perp"] & \mathbb{EED} \arrow[l, bend left, "\mathrm{Rel}"] \\ \mathbb{Fob} \arrow[u, "U"] \arrow[r, bend left, "\mathrm{HmI}"] \arrow[r, phantom, "\perp"] & \mathbb{BHD} \arrow[u, "U"'] \arrow[l, bend left, "\mathrm{Rel}"] \end{tikzcd}$$

$$\mathrm{HmI}(\mathcal{C}) \overset{\mathrm{def}}{:=} \mathrm{Hom}_{\mathcal{C}}[-, I] : \mathrm{Map}(\mathcal{C}) \longrightarrow \mathrm{Bool}$$

BOTH THE ADJUNCTIONS BECOME EQUIVALENCES WHEN RESTRICTING TO DOCTRINES

WITH COMPREHENSIVE DIAGONALS AND RULE OF UNIQUE CHOICHE

INTRODUCED IN (Maietti, Pasquali, Rosolini 2017)

A FOX THEOREM OF CLASSICAL LOGIC

$$
\begin{tikzcd}
\mathbb{CB} \arrow[rr, bend left, "\mathrm{HmI}"] \arrow[rr, phantom, "\perp"] & & \mathbb{EED} \arrow[ll, bend left, "\mathrm{Rel}"] \\
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\end{tikzcd}
$$

(CALCO 2021)

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PEIRCEAN BICATEGORIES

DEFINITION

A Peircean bicategory consists of a Cartesian bicategory such that PROPOSITIONS A and B hold. Namely,

A) every homset carry a boolean algebra

B) for all maps $f : X \to Y$ and arrows $c : Y \to Z$, $f ; c = \overline{f ; \overline{c}}$

THEOREM $\mathbb{F}\mathbf{ob} \cong \mathbb{P}\mathbb{B}$

Conclusion

- Point-free reasoning: no variables, no quantifiers.
- Purely equational reasoning: replace equals by equals
- Complete axiomatisation: (co) Cartesian $\leftrightarrow$ Linear bicategories
- Proofs as diagram rewrites (link to deep inference)
- Functorial semantics: models are structure preserving maps
- Fox-theorem: equational characterisation of hyperdoctrines
- Function symbols are not separate syntactic entities
- The empty model does not require a special treatment

Future Work

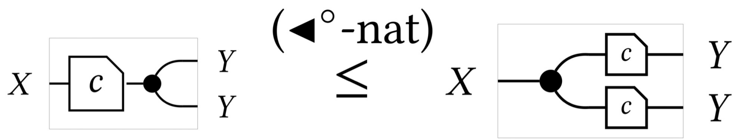
BEYOND FOL: HIGHER ORDER? INTUITIONISTIC? LINEAR?

MULTIRELATION
$X \times Y \longrightarrow \mathbb{N}$

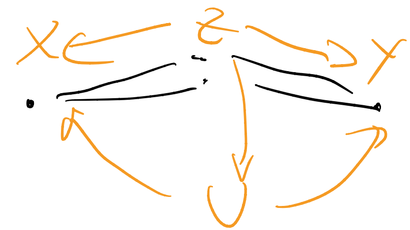
INTUITIONISTIC
LINEAR LOGIC '92

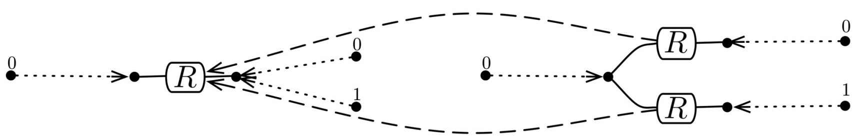
CORRESPONDING ALLEGORICAL NOTIONS?

COMBINATORIAL CHARACTERISATION BY MEANS OF HYPERGRAPHS?



Span(Set)





DEVELOP A PROOF THEORY AND INVESTIGATE THE LINK WITH DEEP INFERENCE