Hybrid Ehrenfeucht-Fraïssé Games

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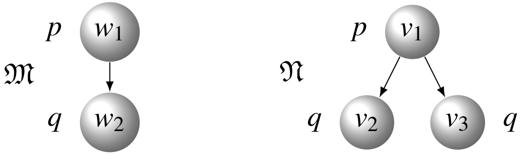
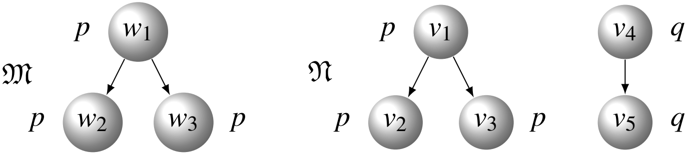
Hybrid Propositional Logic (1)



- $(\mathfrak{M}, w_1) \equiv (\mathfrak{N}, v_1)$ in modal logic with just $\Diamond$
- For $\phi = {\downarrow}z_1 \cdot \Diamond{\downarrow}z_2 \cdot @_{z_1} \Diamond\neg z_2$ in hybrid logic with $\Diamond$, $\downarrow$, and $@$

  $(\mathfrak{M}, w_1) \not\models \phi$ and $(\mathfrak{N}, v_1) \models \phi$

Hybrid Propositional Logic (2)



- $(\mathfrak{M}, w_1) \equiv (\mathfrak{N}, v_1)$ in hybrid logic with $\Diamond$, $\downarrow$, and $@$
- For $\phi = \exists x \cdot @_x\, q$ in hybrid logic with $@$ and $\exists$

$$(\mathfrak{M}, w_1) \not\models \phi \quad \text{and} \quad (\mathfrak{N}, v_1) \models \phi$$

Hybrid Propositional Logic: Signatures and Models

Signatures $\Delta = (\Sigma, \texttt{Prop})$ with $\Sigma = (F, P)$

- nominals F and relations P
- $\Delta[x]$ adds x as new nominal
- "usual" signature morphisms $\chi : (\Sigma_1, \texttt{Prop}_1) \to (\Sigma_2, \texttt{Prop}_2)$

Models $\mathfrak{M} = (W, M) \in \mathsf{Mod}(\Delta)$ over $\Delta = (\Sigma, \texttt{Prop})$

- W first-order structure over Σ
  - interpretations $k^{\mathfrak{M}}$ for nominals and $\lambda^{\mathfrak{M}}$ for relations
- $M : |\mathfrak{M}| \to |\mathsf{Mod}^{\mathsf{PL}}(\texttt{Prop})|$ with $|\mathfrak{M}|$ universe of W
- reduct $\mathfrak{M}|\chi = (W|\chi, M|\chi)$ along $\chi : (\Sigma_1, \texttt{Prop}_1) \to (\Sigma_2, \texttt{Prop}_2)$ with
  - $W|\chi$ first-order reduct of W
  - $M|\chi(w) = M(w)|\chi = \{p \in \texttt{Prop}_1 \mid \chi(p) \in M(w)\}$
- "usual" proposition-preserving homomorphisms $h : (W_1, M_1) \to (W_2, M_2)$

Hybrid Propositional Logic: Sentences

Sentences $\mathrm{Sen}(\Delta)$ over $\Delta = ((F, P), \texttt{Prop})$

$$\phi ::= p \mid k \mid \phi \wedge \phi \mid \neg\phi \mid @_k\, \phi \mid \langle\lambda\rangle\phi \mid {\downarrow}x \cdot \phi_x \mid \exists x \cdot \phi_x$$

p proposition, k nominal, x variable, $\lambda \in P$, $\phi_x \in \mathrm{Sen}(\Delta[x])$

Hybrid language features: retrieve $@$, store $\downarrow$, quantifier $\exists$

- "usual" translation $\chi(\phi)$ for $\chi : \Delta_1 \to \Delta_2$

Hybrid Propositional Logic: Satisfaction

Satisfaction in a pointed model $(\mathfrak{M}, w)$

- "usual" satisfaction for p, $\wedge$, $\neg$
- $(\mathfrak{M}, w) \models k$ if $w = k^{\mathfrak{M}}$
- $(\mathfrak{M}, w) \models @_k\, \phi$ if $(\mathfrak{M}, k^{\mathfrak{M}}) \models \phi$
- $(\mathfrak{M}, w) \models \langle\lambda\rangle\phi$ if $(\mathfrak{M}, v) \models \phi$ for some $v \in \lambda^{\mathfrak{M}}(w)$
  - $\lambda^{\mathfrak{M}}(w) = \{w' \in |\mathfrak{M}| \mid (w, w') \in \lambda^{\mathfrak{M}}\}$
- $(\mathfrak{M}, w) \models {\downarrow}x \cdot \phi$ if $(\mathfrak{M}^{x\leftarrow w}, w) \models \phi$
  - $\mathfrak{M}^{x\leftarrow w}$ unique expansion of $\mathfrak{M}$ to $\Delta[x]$ interpreting x as w
- $(\mathfrak{M}, w) \models \exists x \cdot \phi$ if $(\mathfrak{M}^{x\leftarrow v}, w) \models \phi$ for some $v \in |\mathfrak{M}|$

Satisfaction condition $(\mathfrak{M}, w) \models \chi(\phi)$ iff $(\mathfrak{M}|\chi, w) \models \phi$ holds.

Elementary Equivalence

$(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ elementarily equivalent, $(\mathfrak{M}, w) \equiv (\mathfrak{N}, v)$, if

$$(\mathfrak{M}, w) \models \phi \iff (\mathfrak{N}, v) \models \phi \quad \text{for all } \phi \in \mathrm{Sen}(\Delta)$$

Varies with language fragment $\mathcal{L}$ offering different language features

- modal logic when discarding $@$, $\downarrow$, and $\exists$
- quantifier-free fragment only discarding $\exists$

Goal: Characterising elementary equivalence for different $\mathcal{L}$
in terms of Ehrenfeucht-Fraïssé games

Related Work

Our goal: Parametric handling of different hybrid language fragments

Carlos Areces, Patrick Blackburn, and Maarten Marx. Hybrid logics: characterization, interpolation and complexity. J. Symbolic Logic, 2001

- hybrid bisimulations, back-and-forth systems

Daniel Kernberger and Martin Lange. On the expressive power of hybrid branching-time logics. Theo. Comp. Sci., 2020

- Ehrenfeucht-Fraïssé games for branching time hybrid logics

Samson Abramsky and Dan Marsden. Comonadic semantics for hybrid logic. MFCS 2022.

- Ehrenfeucht-Fraïssé comonad

Ehrenfeucht-Fraïssé Games

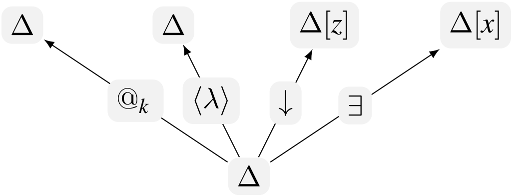
Two-player game between ∃loise and ∀belard

- played on a (complete) gameboard tree tr
  - nodes: finite signatures Δ
  - edges: (labelled) signature morphisms $\Delta \xrightarrow{lb} \Delta'$
    - possible edge labels depend on language fragment $\mathcal{L}$
- game starts with pointed models $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ over $\Delta = root(tr)$
- ∃loise loses if game property not satisfied

  $(\mathfrak{M}, w) \models \phi \iff (\mathfrak{N}, v) \models \phi$ for all basic sentences $\phi \in \mathrm{Sen}_{\mathrm{b}}(\Delta)$

  - $\mathrm{Sen}_{\mathrm{b}}((F, P), \mathtt{Prop})$: nominal $k \in F$, proposition $p \in \mathtt{Prop}$
- if game property holds, ∀belard can move $(\mathfrak{M}, w)$ or $(\mathfrak{N}, v)$ along one of the outgoing edges of tr, ∃loise has to answer

Gameboard Trees



retrieve for nominal k: $\Delta \xrightarrow{@_k} \Delta$ identity (signature morphism)

possibility for relation λ: $\Delta \xrightarrow{\langle\lambda\rangle} \Delta$ identity

store for variable z: $\Delta \xrightarrow{\downarrow} \Delta[z]$ inclusion

exists for variable x: $\Delta \xrightarrow{\exists} \Delta[x]$ inclusion

Moves on a Gameboard Tree (1)

Retrieve

$$\begin{matrix}(\mathfrak{M}, w)\\(\mathfrak{N}, v)\end{matrix}\Delta \xrightarrow{\quad @_k \quad} \Delta \begin{matrix}(\mathfrak{M}, k^{\mathfrak{M}})\\(\mathfrak{N}, k^{\mathfrak{N}})\end{matrix}$$

Possibility for $w \ \lambda^{\mathfrak{M}} \ w_1$ answered by $v \ \lambda^{\mathfrak{N}} \ v_1$

$$\begin{matrix}(\mathfrak{M}, w)\\(\mathfrak{N}, v)\end{matrix}\Delta \xrightarrow{\quad \langle\lambda\rangle \quad} \Delta \begin{matrix}(\mathfrak{M}, w_1)\\(\mathfrak{N}, v_1)\end{matrix}$$

Moves on a Gameboard Tree (2)

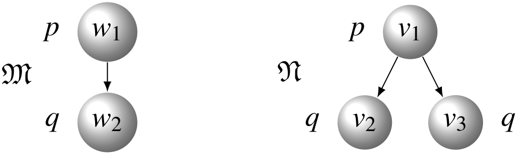
Store

$$\begin{matrix}(\mathfrak{M}, w)\\(\mathfrak{N}, v)\end{matrix}\ \Delta \longrightarrow \boxed{\downarrow} \longrightarrow \Delta[z]\ \begin{matrix}(\mathfrak{M}^{z \leftarrow w}, w)\\(\mathfrak{N}^{z \leftarrow v}, v)\end{matrix}$$

Exists

$$\begin{matrix}(\mathfrak{M}, w)\\(\mathfrak{N}, v)\end{matrix}\ \Delta \longrightarrow \boxed{\exists} \longrightarrow \Delta[x]\ \begin{matrix}(\mathfrak{M}^{x \leftarrow w_1}, w)\\(\mathfrak{N}^{x \leftarrow v_1}, v)\end{matrix}$$

Game Examples (1)

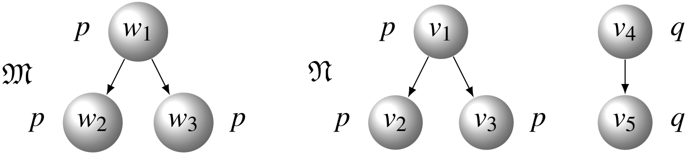


$$(\mathfrak{N}, v_1) \quad (\mathfrak{N}^{z_1 \leftarrow v_1}, v_1) \quad (\mathfrak{N}^{z_1 \leftarrow v_1}, v_2) \quad (\mathfrak{N}^{z_1, z_2 \leftarrow v_1, v_2}, v_2) \quad (\mathfrak{N}^{z_1, z_2 \leftarrow v_1, v_2}, v_1) \quad (\mathfrak{N}^{z_1, z_2 \leftarrow v_1, v_2}, v_3)$$

$$\Delta \;\text{—}\; \downarrow \;\text{—}\; \Delta[z_1] \;\text{—}\; \Diamond \;\text{—}\; \Delta[z_1] \;\text{—}\; \downarrow \;\text{—}\; \Delta[z_1, z_2] \;\text{—}\; @_{z_1} \;\text{—}\; \Delta[z_1, z_2] \;\text{—}\; \Diamond \;\text{—}\; \Delta[z_1, z_2]$$

$$(\mathfrak{M}, w_1) \quad (\mathfrak{M}^{z_1 \leftarrow w_1}, w_1) \quad (\mathfrak{M}^{z_1 \leftarrow w_1}, w_2) \quad (\mathfrak{M}^{z_1, z_2 \leftarrow w_1, w_2}, w_2) \quad (\mathfrak{M}^{z_1, z_2 \leftarrow w_1, w_2}, w_1)$$

Game Examples (2)



$(\mathfrak{N}, v_1)$ $\quad$ $(\mathfrak{N}^{x \leftarrow v_4}, v_1)$ $\quad$ $(\mathfrak{N}^{x \leftarrow v_4}, v_4)$

Δ — $\exists$ — $\Delta[x]$ — $@_x$ — $\Delta[x]$

Fraïssé-Hintikka Theorem

Theorem   Let Δ be a finite signature.

1. For all $(\mathfrak{M}, w)$ over Δ and all gameboard trees tr with $root(tr) = \Delta$, there exists a unique game sentence $\varphi \in \Theta_{tr}$ such that $(\mathfrak{M}, w) \models \varphi$.
2. For all $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ over Δ and all gameboard trees tr with $root(tr) = \Delta$, the following are equivalent:
   (i) $\exists$loise has a winning strategy on tr starting with $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$.
      - $(\mathfrak{M}, w) \approx_{tr} (\mathfrak{N}, v)$

   (ii) There is a unique $\varphi \in \Theta_{tr}$ with $(\mathfrak{M}, w) \models \varphi$ and $(\mathfrak{N}, v) \models \varphi$.
3. If $\mathcal{L}$ is closed under store, then for each sentence ϕ over Δ, there exists a gameboard tree tr with $root(tr) = \Delta$ and a $\Psi_\phi \subseteq \Theta_{tr}$ such that $\phi \leftrightarrow \bigvee \Psi_\phi$ is a tautology.

Game Sentences (1)

Games sentences Θ_{tr} of gameboard tree tr

For $tr = \Delta$:

$$\Theta_\Delta = \{\bigwedge_{\rho \in \mathsf{Sen_b}(\Delta)} \rho^{f(\rho)} \mid f : \mathsf{Sen_b}(\Delta) \to \{0,1\}\}$$

- $\rho^0 = \rho$ and $\rho^1 = \neg\rho$

For $tr = \Delta(\xrightarrow{lb_1} tr_1 \ldots \xrightarrow{lb_n} tr_n)$: Define $S_i \subseteq \mathcal{P}(\Theta_{tr_i})$ and for each $\Gamma \in S_i$ a sentence φ_Γ over Δ; set of game sentences over tr is

$$\Theta_{tr} = \{\varphi_{\Gamma_1} \wedge \cdots \wedge \varphi_{\Gamma_n} \mid \Gamma_1 \in S_1, \ldots, \Gamma_n \in S_n\}$$

Game Sentences (2)

$$\Delta \xrightarrow{@_k} \Delta \quad S_i = \{\{\phi\} \mid \phi \in \Theta_{tr_i}\}$$
$$\varphi_\Gamma = @_k\,\gamma \text{ for } \Gamma = \{\gamma\} \in S_i$$

$$\Delta \xrightarrow{\langle\lambda\rangle} \Delta \quad S_i = \mathcal{P}(\Theta_{tr_i})$$
$$\varphi_\Gamma = (\textstyle\bigwedge_{\gamma\in\Gamma}\langle\lambda\rangle\gamma) \wedge ([\lambda] \bigvee \Gamma)$$

$$\Delta \xrightarrow{\downarrow} \Delta[z] \quad S_i = \{\{\phi\} \mid \phi \in \Theta_{tr_i}\}$$
$$\varphi_\Gamma = {\downarrow}z \cdot \gamma \text{ for } \Gamma = \{\gamma\} \in S_i$$

$$\Delta \xrightarrow{\exists} \Delta[x] \quad S_i = \mathcal{P}(\Theta_{tr_i})$$
$$\varphi_\Gamma = (\textstyle\bigwedge_{\gamma\in\Gamma} \exists x \cdot \gamma) \wedge (\forall x \cdot \bigvee \Gamma)$$

Game Characterisation of Elementary Equivalence

- Gameboard tree construction plays rôle of quantifier rank in first-order logic.
- If $\mathcal{L}$ is closed under possibility, there is no normal form of sentences with first quantifiers, then store, retrieve and Boolean connectives.
  - closure under $\downarrow$ can be replaced by adding identity possibility

Corollary Assume that $\mathcal{L}$ is closed under store. For $\mathfrak{M}$ and $\mathfrak{N}$ over a finite signature Δ the following are equivalent:

1. $(\mathfrak{M}, w) \equiv (\mathfrak{N}, v)$
2. $\exists$loise has a winning strategy for the Ehrenfeucht-Fraïssé game starting with $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$.
   - $(\mathfrak{M}, w) \approx_{tr} (\mathfrak{N}, v)$ for all finite gameboard trees tr

Infinite Ehrenfeucht-Fraïssé Games

Gameboard trees of countably infinite height

- $\exists$loise loses if game property gets violated
- $\exists$loise wins if she can always match any of $\forall$belard's moves

$(\mathfrak{M}, w) \approx_\omega (\mathfrak{N}, v)$

- $\exists$loise has a winning strategy for all gameboard trees of countably infinite height starting with $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$

Goal: Equivalent characterisation of different $\mathcal{L}$ in terms of
infinite Ehrenfeucht-Fraïssé games and back-and-forth systems

Back-and-forth Systems

Basic partial isomorphism $h : \mathfrak{M} \nrightarrow \mathfrak{N}$ bijection between a subset of $|\mathfrak{M}|$ and a subset of $|\mathfrak{N}|$ such that

$$(\mathfrak{M}, w) \models \rho \text{ iff } (\mathfrak{N}, h(w)) \models \rho \quad \text{for all } w \in \mathrm{dom}(h),\ \rho \in \mathsf{Sen}_{\mathrm{b}}(\Delta)$$

Back-and-forth system between $\mathfrak{M}$ and $\mathfrak{N}$ over $\Delta = ((F, P), \mathtt{Prop})$ non-empty family $\mathcal{F}$ of basic partial isomorphisms $\mathfrak{M} \nrightarrow \mathfrak{N}$ satisfying back and forth extension properties depending on the hybrid language features

- @-extension
- $\langle\lambda\rangle$-extension
- $\exists$-extension

Back-and-forth Systems: Extensions

$@$-extension

- for all $h \in \mathcal{F}$ and $k \in F$, there exists $g \in \mathcal{F}$ with $h \subseteq g$ and $k^{\mathfrak{M}} \in \mathrm{dom}(g)$

$\langle\lambda\rangle$-extension for $\lambda \in P$

- "forth": for all $h \in \mathcal{F}$, $w_1 \in \mathrm{dom}(h)$, and $w_2 \in |\mathfrak{M}|$ with $w_1\ \lambda^{\mathfrak{M}}\ w_2$, there exists $g \in \mathcal{F}$ with $h \subseteq g$, $w_2 \in \mathrm{dom}(g)$, and $g(w_1)\ \lambda^{\mathfrak{N}}\ g(w_2)$;
- "back": analogous

$\exists$-extension

- "forth": for all $h \in \mathcal{F}$ and $w \in |\mathfrak{M}|$, there exists $g \in \mathcal{F}$ with $h \subseteq g$ and $w \in \mathrm{dom}(g)$;
- "back": analogous

Back-and-forth Systems vs. Partial Isomorphisms

Back-and-forth equivalence

- $\mathfrak{M} \equiv_{\mathcal{F}} \mathfrak{N}$ if $\mathcal{F}$ back-and-forth system between $\mathfrak{M}$ and $\mathfrak{N}$
- $(\mathfrak{M}, w) \equiv_{\mathcal{F}} (\mathfrak{N}, v)$ if $\mathfrak{M} \equiv_{\mathcal{F}} \mathfrak{N}$ such that $h(w) = v$ for some $h \in \mathcal{F}$

Partial isomorphism $h : \mathfrak{M} \nrightarrow \mathfrak{N}$ basic partial isomorphism with

$$w_1 \ \lambda^{\mathfrak{M}} \ w_2 \text{ iff } h(w_1) \ \lambda^{\mathfrak{N}} \ h(w_2) \quad \text{for all } \lambda \in P, w_1, w_2 \in \mathrm{dom}(h)$$

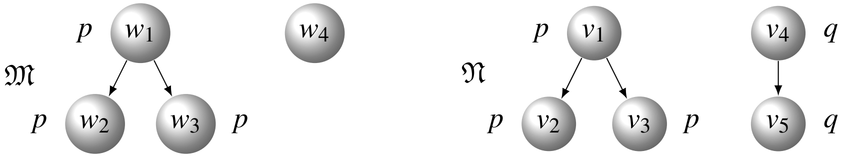
Lemma Any basic partial isomorphism belonging to a back-and-forth system closed under possibility-extensions is a partial isomorphism.

Back-and-forth Systems vs. Ehrenfeucht-Fraïssé Games

Theorem Assume that $\mathcal{L}$ is closed under store and that $\mathcal{L}$ is closed under retrieve if it is closed under existential quantifiers. For $\mathfrak{M}$ and $\mathfrak{N}$ over a finite signature Δ the following are equivalent:

1. $(\mathfrak{M}, w) \approx_\omega (\mathfrak{N}, v)$
   - $\exists$loise has a winning strategy for the countably infinite Ehrenfeucht-Fraïssé game starting with $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$.
2. $(\mathfrak{M}, w) \equiv_\mathcal{F} (\mathfrak{N}, v)$
   - There is a back-and-forth system $\mathcal{F}$ between $\mathfrak{M}$ and $\mathfrak{N}$ which contains a basic partial isomorphism sending w to v.

Back-and-forth Systems vs. Ehrenfeucht-Fraïssé Games: Example



For $\mathcal{L}$ without nominals and @, but containing $\exists$:

$$(\mathfrak{M}, w_1) \approx_\omega (\mathfrak{N}, v_1) \quad \text{and} \quad (\mathfrak{M}, w_1) \not\equiv_{\mathcal{F}} (\mathfrak{N}, v_1)$$

- non-reachable states cannot be compared, but there is no "forth" $\exists$-extension for w_4

From Back-and-forth to Ehrenfeucht-Fraïssé

Lemma Let $\mathfrak{M} \equiv_{\mathcal{F}} \mathfrak{N}$ and $w_1, \ldots, w_n, w_{n+1} \in |\mathfrak{M}|$ and $v_1, \ldots, v_n, v_{n+1} \in |\mathfrak{N}|$ with some $h \in \mathcal{F}$ such that $h(w_i) = v_i$ for all $i \in \{1, \ldots, n+1\}$. Then $\mathfrak{M}^{z_1,\ldots,z_n \leftarrow w_1,\ldots,w_n} \equiv_{\mathcal{F}'} \mathfrak{N}^{z_1,\ldots,z_n \leftarrow v_1,\ldots,v_n}$ for some back-and-forth system $\mathcal{F}'$.

Given that $(\mathfrak{M}, w) \equiv_{\mathcal{F}} (\mathfrak{N}, v)$ with $h \in \mathcal{F}$ such that $h(w) = v$, construct winning strategy for ∃loise along back-and-forth equivalent pairs of pointed models using the extension properties, e. g.

- $\Delta \xrightarrow{\downarrow} \Delta[z]$ By lemma, $\mathfrak{M}^{z \leftarrow w} \equiv_{\mathcal{F}'} \mathfrak{N}^{z \leftarrow v}$; thus $(\mathfrak{M}^{z \leftarrow w}, w) \equiv_{\mathcal{F}'} (\mathfrak{N}^{z \leftarrow v}, v)$.
- $\Delta \xrightarrow{\langle\lambda\rangle} \Delta$ ∀belard has chosen $w' \in |\mathfrak{M}|$ with $w \; \lambda^{\mathfrak{M}} \; w'$; by forth $\langle\lambda\rangle$-exten., ex. $g \in \mathcal{F}$ with $h \subseteq g$, $w' \in \mathrm{dom}(g)$, $g(w) \; \lambda^{\mathfrak{N}} \; g(w')$.
  ∃loise can choose $v' = g(w')$ s. t. $(\mathfrak{M}, w') \equiv_{\mathcal{F}} (\mathfrak{N}, v')$.

From Ehrenfeucht-Fraïssé to Back-and-forth (1)

For $n > 0$, let $w_1, \ldots, w_n \in |\mathfrak{M}|$, $v_1, \ldots, v_n \in |\mathfrak{N}|$ s. t. $(\mathfrak{M}^{z_1,\ldots,z_i \leftarrow w_1,\ldots,w_i}, w_{i+1}) \approx_\omega (\mathfrak{N}^{z_1,\ldots,z_i \leftarrow v_1,\ldots,v_i}, v_{i+1})$ for all $1 \leq i \leq n-1$.

- For new variable z_n, move along $\Delta[z_1, \ldots, z_{i-1}] \xrightarrow{\downarrow} \Delta[z_1, \ldots, z_{i-1}, z_i]$ yields $(\mathfrak{M}^{z_1,\ldots,z_i \leftarrow w_1,\ldots,w_i}, w_i) \approx_\omega (\mathfrak{N}^{z_1,\ldots,z_i \leftarrow v_1,\ldots,v_i}, v_i)$ for all $1 \leq i \leq n$.
- Then $h : \mathfrak{M} \nrightarrow \mathfrak{N}$ with $h(w_i) = v_i$ for all $1 \leq i \leq n$:
  - Injectivity: $w_i = w_j$ iff $(\mathfrak{M}^{z_1,\ldots,z_j \leftarrow w_1,\ldots,w_j}, w_j) \models z_i$
    iff $(\mathfrak{N}^{z_1,\ldots,z_j \leftarrow v_1,\ldots,v_j}, v_j) \models z_i$ iff $v_i = v_j$.
  - Satisfaction of basic sentences: satisfaction condition

From Ehrenfeucht-Fraïssé to Back-and-forth (2)

- h can be extended to another basic partial isomorphism $h \cup \{w \mapsto v\}$ according to back-and-forth extensions such that $(\mathfrak{M}^{z_1,\ldots,z_n \leftarrow w_1,\ldots,w_n}, w) \approx_\omega (\mathfrak{N}^{z_1,\ldots,z_n \leftarrow v_1,\ldots,v_n}, v)$, e. g.,

  $\langle\lambda\rangle$-extension Let $w_n \; \lambda^{\mathfrak{M}} \; w$ hold. Consider move along

  $\Delta[z_1, \ldots, z_n] \xrightarrow{\langle\lambda\rangle} \Delta[z_1, \ldots, z_n]$ s. t.
  $(\mathfrak{M}^{z_1,\ldots,z_n \leftarrow w_1,\ldots,w_n}, w) \approx_\omega (\mathfrak{N}^{z_1,\ldots,z_n \leftarrow v_1,\ldots,v_n}, v)$.
  Then $h \cup \{w \mapsto v\} : \mathfrak{M} \nrightarrow \mathfrak{N}$ by checking injectivity and satisfaction of basic sentences.

Given that $(\mathfrak{M}, w_1) \approx_\omega (\mathfrak{N}, v_1)$, start with basic partial isomorphism $h : \mathfrak{M} \nrightarrow \mathfrak{N}$ with $h(w_1) = v_1$ and extend it an arbitrary number of times.

Reachable and Image-finite Models (1)

$\mathfrak{M}$ reachable if all states reachable

- $w \in |\mathfrak{M}|$ reachable if $w \in (\bigcup_{\lambda \in P} \lambda^{\mathfrak{M}})^*(k^{\mathfrak{M}})$ for some nominal k

$\mathfrak{M}$ image-finite if $\lambda^{\mathfrak{M}}(w)$ finite for each $w \in |\mathfrak{M}|$ and all $\lambda \in P$

Lemma Let $\mathfrak{M}$ and $\mathfrak{N}$ be image-finite over Δ such that $(\mathfrak{M}, w) \equiv (\mathfrak{N}, v)$ for some $w \in |\mathfrak{M}|$, $v \in |\mathfrak{N}|$. Then:

1. w and v have the same number of λ-successors, for all λ in Δ.
2. For all λ in Δ and all $w_1 \in |\mathfrak{M}|$ with $w \; \lambda^{\mathfrak{M}} \; w_1$ there exists a $v_1 \in |\mathfrak{N}|$ with $v \; \lambda^{\mathfrak{M}} \; v_1$ and $(\mathfrak{M}, w_1) \equiv (\mathfrak{N}, v_1)$.

Reachable and Image-finite Models (2)

Consider quantifier-free fragment

Theorem Let $\mathfrak{M}$ and $\mathfrak{N}$ be reachable over the finite signature Δ with at least one nominal.

1. If $\mathfrak{M}$ and $\mathfrak{N}$ are countable and $\mathfrak{M} \equiv_{\mathcal{F}} \mathfrak{N}$, then $\mathfrak{M} \cong \mathfrak{N}$.
2. If $\mathfrak{M}$ and $\mathfrak{N}$ are image-finite and $(\mathfrak{M}, k^{\mathfrak{M}}) \equiv (\mathfrak{N}, k^{\mathfrak{N}})$ for all nominals k, then $\mathfrak{M} \equiv_{\mathcal{F}} \mathfrak{N}$ for some back-and-forth system $\mathcal{F}$.

Proof idea

1. Consider enumeration of states along possibilities and nominals; construct ascending chain of partial isomorphims.
2. Show that ∃loise has a winning strategy in any Ehrenfeucht-Fraïssé game starting from some nominal: ∃loise has a winning strategy when starting in $(\mathfrak{M}, w) \equiv (\mathfrak{N}, v)$ by previous lemma.

Reachable and Image-finite Models (3)

Consider quantifier-free fragment

Corollary Let $\mathfrak{M}$ and $\mathfrak{N}$ be reachable and image-finite over the finite signature Δ with at least one nominal. If $(\mathfrak{M}, k^{\mathfrak{M}}) \equiv (\mathfrak{N}, k^{\mathfrak{N}})$ for all nominals k, then $\mathfrak{M} \cong \mathfrak{N}$.

- image-finiteness necessary, like in modal logic
- also applicable to rooted models (without nominals)

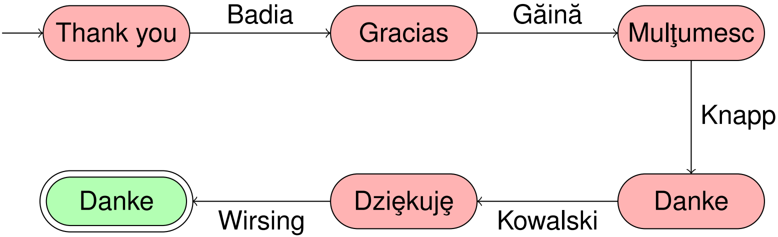
Conclusions and Future Work

Ehrenfeucht-Fraïssé games for hybrid propositiona logic

- parametric in the language features using gameboard trees
- finite and countably infinite versions
- characterisation of elementary equivalence and back-and-forth systems

Connection to bisimulations

- G. Badia, D. Găină, A. K., T. Kowalski, M. Wirsing. A Modular Bisimulation Characterisation for Fragments of Hybrid Logic. Submitted, 2024.


Thank you
Badia
Gracias
Găină
Mulţumesc
Knapp
Danke
Kowalski
Dziękuję
Wirsing
Danke