Hennessy-Milner Theorems via Galois Connections

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Motivation

Hennessy-Milner theorems connect behavioural equivalences with modal logics. Given two states $x_1, x_2 \in X$ and formulas φ:

$$x_1 \sim x_2 \iff \forall \varphi\colon (x_1 \models \varphi \Leftrightarrow x_2 \models \varphi)$$

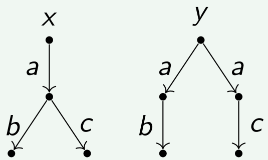
There is a metric analogue, where d is a pseudo-metric on the state space and formulas φ evaluate to real-valued predicates $[\![\varphi]\!]\colon X \to [0,1]$

$$d(x_1, x_2) = \bigvee_{\varphi} \big| [\![\varphi]\!](x_1) - [\![\varphi]\!](x_2) \big|$$

Given a behavioural equivalence, the aim is typically to determine a modal logic that characterizes this equivalence
$\leadsto$ linear-time/branching-time spectrum [van Glabbeek], coalgebraic modal logics

Motivation

Example



x, y are not bisimilar ($x \not\sim y$)

They are distinguished by $\varphi = \Diamond_a(\Diamond_b\, true \wedge \Diamond_c\, true)$ where $x \models \varphi$, $y \not\models \varphi$.

Motivation

Our contributions:

- Hennessy-Milner theorems can be obtained from the fact that least fixpoints are preserved by left adjoints (of a Galois connection).
- Rather than starting with the definition of a behavioural equivalence, we go the other way and derive fixpoint equations for behavioural equivalences/metrics from the modal logics. (Including compositionality results.)
- We obtain (new) fixpoint equations for decorated trace metrics.

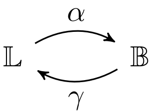
Galois Connection

Definition

Let $\mathbb{L}$, $\mathbb{B}$ be two complete lattices with order $\sqsubseteq$. A Galois connection from $\mathbb{L}$ to $\mathbb{B}$ is a pair $\alpha \dashv \gamma$ of monotone functions such that

$$\alpha(\ell) \sqsubseteq m \iff \ell \sqsubseteq \gamma(m),$$

for all $\ell \in \mathbb{L}$, $m \in \mathbb{B}$.



Intuition in our case: $\mathbb{L}$ – logical universe, $\mathbb{B}$ – behaviour universe

General setting

$$\mathsf{log} \circlearrowleft \mathbb{L} \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} \mathbb{B} \circlearrowright \mathsf{beh}$$

$$\mathsf{beh} = \alpha \circ \mathsf{log} \circ \gamma$$

Compatibility

Let $\mathsf{log}, c \colon \mathbb{L} \to \mathbb{L}$ be two monotone endo-functions on a lattice $\mathbb{L}$.
We call log *c-compatible* whenever $\mathsf{log} \circ c \sqsubseteq c \circ \mathsf{log}$.

Compatibility: concept borrowed from up-to techniques

Theorem

Let $\alpha\colon \mathbb{L} \to \mathbb{B}$, $\gamma\colon \mathbb{B} \to \mathbb{L}$ be a Galois connection and let $\mathsf{log}\colon \mathbb{L} \to \mathbb{L}$, $\mathsf{beh}\colon \mathbb{B} \to \mathbb{B}$ (both monotone).

1. Then $\alpha \circ \mathsf{log} = \mathsf{beh} \circ \alpha$ implies $\alpha(\mu\,\mathsf{log}) = \mu\,\mathsf{beh}$.
2. Let $c = \gamma \circ \alpha$ be the closure operator of the Galois connection and let $\mathsf{beh} = \alpha \circ \mathsf{log} \circ \gamma$.
   Then c-compatibility of log implies $\alpha(\mu\,\mathsf{log}) = \mu\,\mathsf{beh}$.

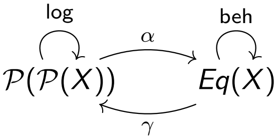
μ: least fixpoint operator

This theorem is well-known and goes back to work of Cousot & Cousot on abstract interpretation.

Bisimilarity

We instantiate this framework and start with the simplest case: bisimilarity on labelled transition systems (with state space X).

Bisimilarity



$Eq(X)$: set of all equivalences on X, ordered by $\supseteq$

$$\begin{aligned} \alpha(\mathcal{S}) &= \{(x_1, x_2) \in X \times X \mid \forall S \in \mathcal{S}\colon (x_1 \in S \Leftrightarrow x_2 \in S)\} \\ \gamma(R) &= \{S \subseteq X \mid \forall (x_1, x_2) \in R\colon (x_1 \in S \Leftrightarrow x_2 \in S)\} \end{aligned}$$

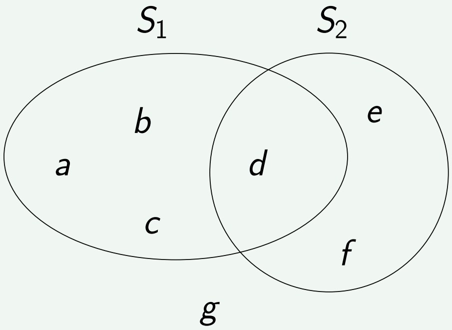
If $\mathrm{log} \circ c \subseteq c \circ \mathrm{log}$ (where $c = \gamma \circ \alpha$) and $\mathrm{beh} = \alpha \circ \mathrm{log} \circ \gamma$:

$$\alpha(\mu\, \mathrm{log}) = \mu\, \mathrm{beh}$$

(μ: least fixpoint. Contravariance!) This is the Hennessy-Milner theorem (logical equivalence = behavioural equivalence).

Bisimilarity

Obtaining $\alpha(\{S_1, S_2\})$ for $S_1, S_2 \subseteq X = \{a, b, c, d, e, f, g\}$



Bisimilarity

- Logic function:

$$\log(\mathcal{S}) = \bigcup_{a\in A} \Diamond_a[cl_f(\mathcal{S})]$$

  - cl_f closes a set of sets under all finite boolean operations (empty conjunction: *true*, empty disjunction: *false*)
  - $\Diamond_a(S) = \{x \in X \mid \exists y \in S \colon x \xrightarrow{a} y\}$
- Closure: $c = \alpha \circ \gamma$ closes a set of sets under all boolean operators
  log is compatible with c if transition system is *finitely branching*
- Behaviour function: for $R \in Eq(X)$

$$\mathrm{beh}(R) = \alpha(\log(\gamma(R))) =$$
$$\{(x_1, x_2) \mid \forall y_1 \colon x_1 \xrightarrow{a} y_1\ \exists y_2 \colon x_2 \xrightarrow{a} y_2 \land (y_1, y_2) \in R \land \forall y_2 \colon x_2 \xrightarrow{a} y_2\ \exists y_1 \colon x_1 \xrightarrow{a} y_1 \land (y_1, y_2) \in R\}$$

Trace Metrics

Trace equivalence can be generalized to trace metrics [de Alfaro, Faella, Stoelinga] [Fahrenberg, Legay] that measures the distance between the sets of traces originating from two states.

Useful for systems with quantitative information (probabilities, weights, etc.) where behavioural equivalence is too strict.

Here: we generalize the trace inclusion preorder to a directed trace metrics.

First step: Extend transition systems with a metric $d_A \colon A \times A \to [0, 1]$ on the label set A.

Trace Metrics

Preliminaries on metrics

- *DPMet*(Y): set of all directed pseudo-metrics on Y, i.e., functions $d\colon Y \times Y \to [0,1]$ such that
  - $d(y,y) = 0$ for all $y \in Y$
  - $d(y_1,y_3) \le d(y_1,y_2) + d(y_2,y_3)$ (triangle inequality) for all $y_1, y_2, y_3 \in Y$.
  - not necessarily symmetric ($d(y_1,y_2) = d(y_2,y_1)$)
- Directed pseudometric space: set Y with a directed pseudo-metric d
- Non-expansive functions between pseudometric spaces (Y, d_Y), (Z, d_Z): mapping $f\colon Y \to Z$ with $d_Z(f(y_1), f(y_2)) \le d_Y(y_1,y_2)$ for all $y_1, y_2 \in Y$.

Addition and subtraction are modified to stay within [0,1]

Trace Metrics

Directed Hausdorff metric

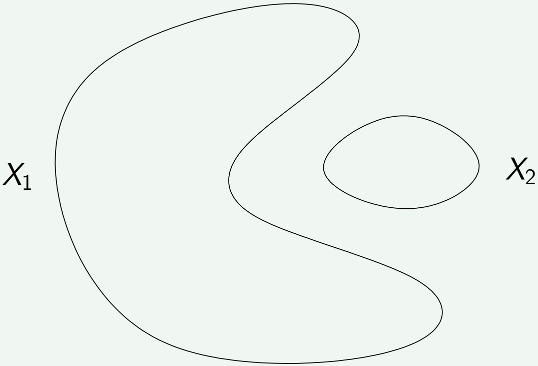
Lifting a directed metric space (X, d) to $(\mathcal{P}(X), d^H)$: let $X_1, X_2 \subseteq X$:

$$d^H(X_1, X_2) = \max_{x_1 \in X_1} \min_{x_2 \in X_2} d(x_1, x_2)$$

- For each element $x_1 \in X_1$ take the closest element $x_2 \in X_2$ and measure the distance $d(x_1, x_2)$
- Take the maximum of all such distances.

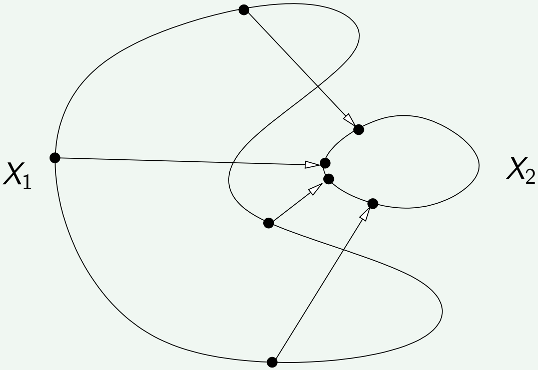
Trace Metrics

Example: Directed Hausdorff metric



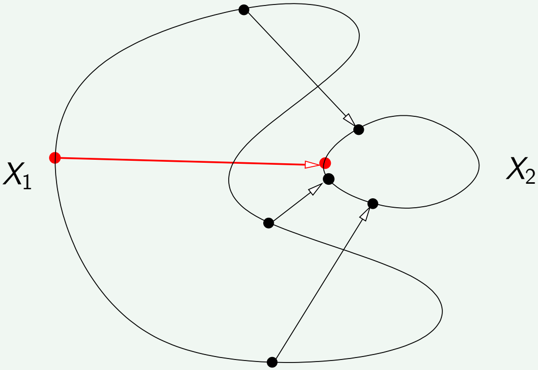
Trace Metrics

Example: Directed Hausdorff metric



Trace Metrics

Example: Directed Hausdorff metric



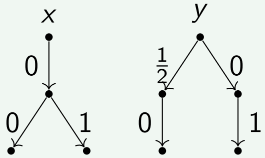
Trace Metrics

Trace Distance of two states x, y

- Let $Tr(x) \subseteq A^*$ be the set of finite traces of x.
- The distance of two traces σ_1, σ_2 is defined as
  - $d_{\mathrm{Tr}}(\sigma_1, \sigma_2) = 1$ if $|\sigma_1| \neq |\sigma_2|$
  - $d_{\mathrm{Tr}}(\varepsilon, \varepsilon) = 0$
  - $d_{\mathrm{Tr}}(a_1\sigma_1', a_2\sigma_2') = \max\{d_A(a_1, a_2), d_{\mathrm{Tr}}(\sigma_1', \sigma_2')\}$ (sup-metric)
- Given two states x, y:

$$d(x, y) = (d_{\mathrm{Tr}})^H(Tr(x), Tr(y))$$

Trace Metrics



For simplicity we restrict to traces of length 2:

$$d(x,y) = (d_{\mathrm{Tr}})^H(\ \{00,\quad 01\}\quad \{\tfrac{1}{2}\,0,\quad 01\}\ )$$

Hence: $d(x,y) = \frac{1}{2}$.

Trace Metrics

We apply our receipe and use the following Galois connection:

$$\text{log} \circlearrowright (\mathcal{P}([0,1]^X), \subseteq) \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} (\mathit{DPMet}(\mathcal{P}(X)), \leq) \circlearrowleft \text{beh}$$

$$\alpha(\mathcal{F})(X_1, X_2) = \bigvee_{f \in \mathcal{F}} (\tilde{f}(X_1) - \tilde{f}(X_2))$$

$$\gamma(d) = \{f \in [0,1]^X \mid \tilde{f} \text{ is non-expansive wrt. } d\}$$

where $\tilde{f} \colon \mathcal{P}(X) \to [0,1]$ $\qquad \tilde{f}(X') = \bigvee_{x \in X'} f(x)$
$(f \colon X \to [0,1])$

Trace Metrics

Logic function:

$$\text{Modality:} \qquad \bigcirc_a f(x) = \bigvee\{\overline{D_a}(b) \wedge f(x') \mid x \xrightarrow{b} x'\}$$

where $a \in A$, $f \colon X \to [0,1]$, $\overline{D_a}(b) = 1 - d_A(b,a)$.

$$\log(\mathcal{F}) = \bigcup_{a \in A} \bigcirc_a [cl^{\mathrm{sh}}(\mathcal{F})] \cup \{1\},$$

where cl^{sh} closes a set of functions under constant shifts ($f \mapsto f + c, f - c,\ c \in [0,1]$).

The logic function is compatible with the closure of the Galois connection (shifts are needed for compatibility).

Trace Metrics

Completeness:

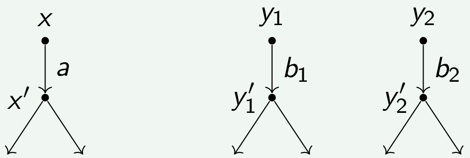
In order to convince ourselves that the logic is complete, we construct a distinguishing formula φ.

To obtain the trace distance of states x, y, take the trace $a_1 \dots a_n \in Tr(x)$ of x that is farthest from any trace in $Tr(y)$.

Define $\varphi = \bigcirc_{a_1} \cdots \bigcirc_{a_n} 1$

Trace Metrics

Fixpoint function/equation (special case): $\mathsf{beh} = \alpha \circ \log \circ \gamma$



$$
\begin{aligned}
& \mathsf{beh}(d)(\{x\},\{y_1,y_2\}) \\
= \; & (d_A(a,b_1) \wedge d_A(a,b_2)) \vee \big(d_A(a,b_1) \wedge d(\{x'\},\{y_2'\})\big) \\
& \quad \vee \big(d_A(a,b_2) \wedge d(\{x'\},\{y_2'\})\big) \vee d(\{x'\},\{y_1',y_2'\})
\end{aligned}
$$

This result depends on the fact that $([0,1],\leq)$ is a distributive lattice.

Further Results

We can also handle . . .

- Preorders
- Behavioural metrics
- Decorated trace equivalences and metrics (completed traces, readiness, failure, etc.)

on labelled transition systems respectively metric transition systems.

Coalgebra & Fibrations

Future work: generalization for coalgebras living in Eilenberg-Moore categories (with Lutz Schröder, Jonas Forster, Paul Wild)

It is our aim to show that the following three notions induce the same metric:

- behavioural metrics obtained in the graded monad setting
- graded modal logics
- least solution of a fixpoint equation defined on the determinized coalgebra

Conclusion

Related Work

- Fahrenberg, Legay, Thrane: Characterization of the metric linear-time/branching-time spectrum via games. Does not treat logics and fixpoint equations for trace metrics are different.
- Klin (e.g. in Klin's PhD thesis): different handling of the closure, does not treat behavioural metrics.
- Dual adjunction: functor on the "logic universe" characterizes the *syntax* of the logics rather than the *semantics*. Fibrational setup deviates from [Kupke, Rot].
- Approximating family [Komorida, Katsumata, Kupke, Rot, Hasuo]: related to our notion of compatibility.