Lax Extensions, Behavioural Metrics and Modal Logic

Paul Wild and Lutz Schröder

Friedrich-Alexander-Universität Erlangen-Nürnberg

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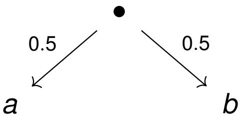
Introduction

- Behavioural metrics offer fine-grained notion of process comparison
  - Fuzzy, weighted, metric transition systems
  - Markov chains
- Coalgebraic approach via metric liftings of functors (Baldan et al.)
- Stronger notion: lax extensions
  - more precisely nonexpansive lax extensions
- Generic Kantorovich, Wasserstein liftings / extensions
- Every lax extension is Kantorovich
- Consequence: Every behavioural metric admits a characteristic logic

Behavioural metrics

E.g. probabilistic systems:



– not bisimilar, but „close“

- Behavioural distance 0.1 under standard definitions (*earth movers metric*)

Coalgebra

Coalgebras = generic reactive systems

- Set X of states
- Transition structure $X \to FX$
- Functor F is the type of the system.
- E.g. $F = \mathcal{P}$: Non-deterministic branching

Quantitative Systems as Coalgebras

- Markov chains (with deadlocks): e.g. $X \to \mathcal{D}(X+1)$
- Fuzzy transition systems: e.g. $X \to (X \to [0,1]) \times (\mathrm{At} \to [0,1])$
- Metric transition systems: e.g. $X \to \mathcal{P}(S \times X)$, $X \to S \times \mathcal{P}(X)$
- Markov decision processes: e.g. $X \to \mathcal{P}(\mathcal{D}X + 1)$, $X \to C(X+1)$
  - $CX =$ convex sets of distributions on X

Fuzzy Relations

Fuzzy relation $R\colon X \nrightarrow Y$ = map $R\colon X \times Y \to [0,1]$

- $0 = \text{true}$
- $R^\circ\colon Y \nrightarrow X$
- For $R\colon X \nrightarrow Y$, $S\colon Y \nrightarrow Z$: $R;S\colon X \nrightarrow Z$,

$$(R;S)(x,z) = \inf_{y \in Y} R(x,y) \oplus S(y,z)$$

- For $f\colon X \to Y$, $\mathrm{Gr}(f)(x,y) = 0$ if $f(x) = y$, 1 otherwise; $\Delta = \mathrm{Gr}(\mathrm{id})$

$d\colon X \nrightarrow X$ pseudometric iff

$$d = d^\circ \qquad d \leq d;d \qquad d \leq \Delta$$

Lax Extensions

Lax extension L for functor F:

$$R\colon X \nrightarrow Y \mapsto LR\colon FX \nrightarrow FY$$

such that

$$\begin{array}{ll} \text{(L0)} & L(R^\circ) = (LR)^\circ \\ \text{(L1)} & R_1 \leq R_2 \Rightarrow LR_1 \leq LR_2 \\ \text{(L2)} & L(R;S) \leq LR;LS \\ \text{(L3)} & L\mathrm{Gr}_f \leq \mathrm{Gr}_{Ff} \end{array}$$

Then L preserves pseudometrics. L non-expansive if

$$\text{(L4)} \quad L\Delta_{\varepsilon,X} \leq \Delta_{\varepsilon,FX}$$

where $\Delta_\varepsilon(x,x) = \varepsilon$. Then $L\colon (X \nrightarrow Y) \to (FX \nrightarrow FY)$ non-expansive

Kantorovich Liftings

Recall: (fuzzy) predicate lifting for F = natural transformation

$$\lambda_X \colon [0,1]^X \to [0,1]^{FX}$$

Given set Λ of predicate liftings, closed under duals, have Kantorovich, erm, lifting

$$K_\Lambda R(t,s) = \sup\{\lambda_X(f)(t) - \lambda_Y(g)(s) \mid \lambda \in \Lambda, (f,g) \ R\text{-nonexpansive}\}$$

where (f,g) R-non-expansive if $f(x) - g(y) \leq R(x,y)$ for all x, y.

Example: $\lambda = \mathbb{E}$ (expectation) gives (on metrics) standard Kantorovich(-Wasserstein) lifting of $\mathcal{D}$.

Wasserstein Liftings

Again, given set Λ of predicate liftings, have Wasserstein lifting

$$W_\Lambda R(t,s) = \sup_{\lambda\in\Lambda} \inf\{\lambda_{X\times Y}(R)(u) \mid F\pi_1(u) = t, F\pi_2(u) = s\}$$

(e.g. $F = \mathcal{D}$: u joint distribution with marginals t, s)

Examples:

- $F = \mathcal{P}$, $\lambda_X(f)(A) = \sup_{x\in X} f(x)$ gives standard Hausdorff lifting

$$HR(A,B) = \max(\sup_{x\in A}\inf_{y\in B} R(x,y),\ \sup_{y\in B}\inf_{x\in A} R(x,y)).$$

- $F = \mathcal{C}$ (convex powerset), $\lambda_X(f)(A) = \sum_{\mu\in A}\mathbb{E}_\mu(f)$ gives (restriction of) composition of Hausdorff and Kantorovich-Wasserstein.

Behavioural Metrics via Lax Extensions

$R\colon C \nrightarrow D$ is an **L-bisimulation** between F-coalgebras $\gamma\colon C \to FC$, $\delta\colon D \to FD$ if $R(x,y) \geq LR(\gamma(x),\delta(y))$ for all x,y:

$$\begin{array}{ccc} x & \overset{R}{\underset{\geq}{\longrule}} & y \\ \Big\downarrow & & \Big\downarrow \\ \gamma(x) & \xrightarrow[LR]{} & \delta(y) \end{array}$$

L-Behavioural distance = least L-bisimulation

Every Lax Extension is Kantorovich

Finitary F has *functor presentation* by finitary operations $\sigma_X \colon X^n \to FX$. These induce Moss liftings $\mu_X^\sigma \colon ([0,1]^X)^n \to [0,1]^{TX}$

$$\mu_X^\sigma(f_1,\dots,f_n)(t) = (L \ni_X)(\sigma_{[0,1]^X}(f_1,\dots,f_n),t),$$

where $\ni \colon [0,1]^X \nrightarrow X$. Then

$$L = K_\Lambda$$

for Λ = Moss liftings and their duals.

Moss liftings non-expansive iff L non-expansive

What does that mean?

Recall: Set Λ of crisp predicate liftings separating if for $t, s \in FX$,

$$t \neq s \Rightarrow \exists \lambda \in \Lambda, A \in 2^X.\, \lambda_X(A)(t) \neq \lambda_X(A)(s)$$

Similarly, $L \leq K_\Lambda$ iff

$$LR(t,s) > \varepsilon \Rightarrow \exists \lambda \in \Lambda, (f,g)\ R\text{-non-expansive}.\, \lambda_X(f)(t) - \lambda_X(g)(s) > \varepsilon$$

(Expressiveness)

Characteristic Logics

Quantitative coalgebraic modal logic:

$$\phi, \psi ::= c \mid \phi \ominus c \mid \neg\phi \mid \phi \wedge \psi \mid \lambda\phi$$

- Induces logical distance
- Hennessy Milner property: Logical distance = behavioural distance
- König/Mika-Michalski CONCUR 2019: This holds for Kantorovich distances approximable in ω steps.
- Wild/Schröder CONCUR 2020: These conditions hold if F is finitarily separable and the predicate liftings are non-expansive.
- Summing up, every non-expansive lax extension admits a characteristic modal logic.

Conclusions

- (Fuzzy) lax extensions induce behavioural distance via bisimulations
- Every lax extension is Kantorovich
  - quantitative variant of known two-valued results:
    - Every finitary functor admits a separating set of predicate liftings (Schröder 2006)
    - Every two-valued lax extension of a finitary functor admits a separating set of monotone predicate liftings (Kurz/Leal 2009, Marti/Venema 2012).
- Key property: non-expansiveness
- Non-expansive lax extensions admit characteristic modal logics

Future Work

- Generalize:
  - Quantale-valued metrics
  - Asymmetric distances (hemimetrics) (quantitative simulation)
- Quantitative (not ‘quantified’...) coalgebraic van Benthem theorem at FoSSaCS (Tue 3pm)
- Spectra of behavioural distances
  - Graded monads