A Diagrammatic Algebra of Linear and Concurrent Systems

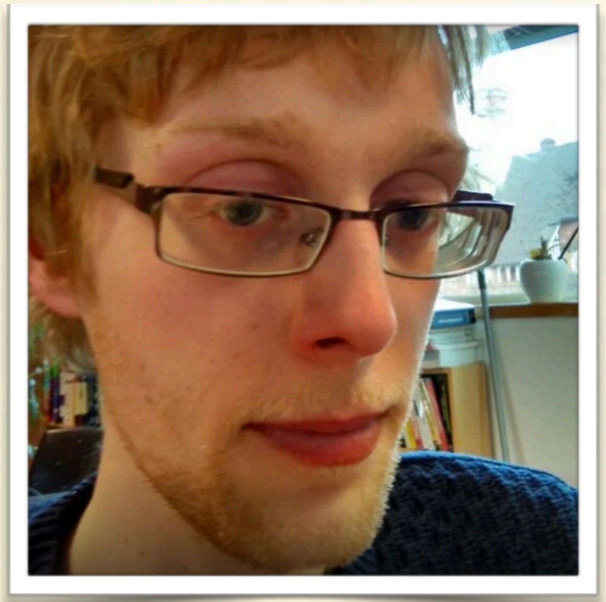
Fabio Zanasi
University College London

IFIP WP1.3 Meeting, Prague
April 2019

Collaborators

Filippo
Bonchi

U. Pisa



Josh
Holland

U. Southampton

Robin
Piedeleu

UCL

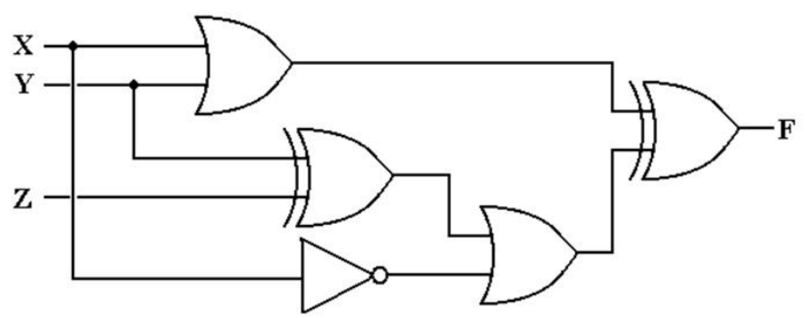
Pawel
Sobocinski

U. Southampton

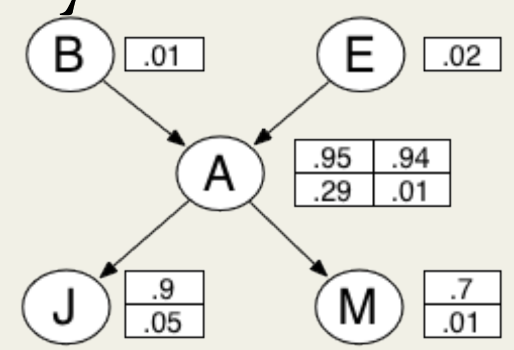
Introduction

Component-Based Systems

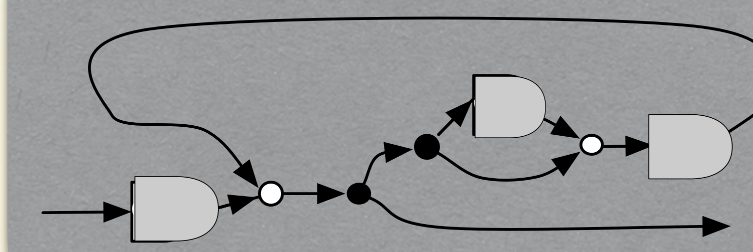
Digital Circuits



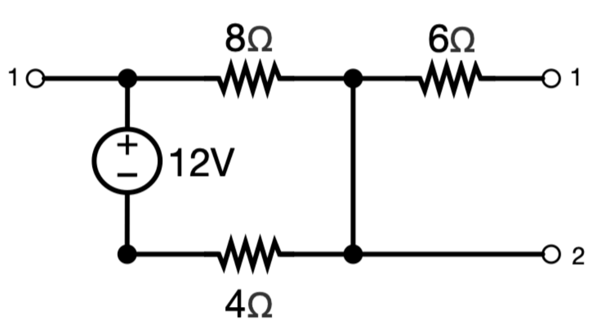
Bayesian Networks



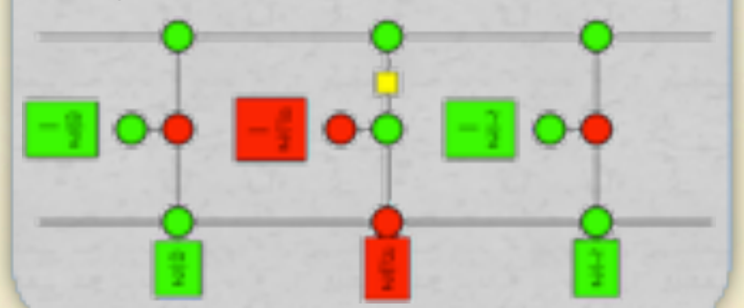
Signal Flow Graphs



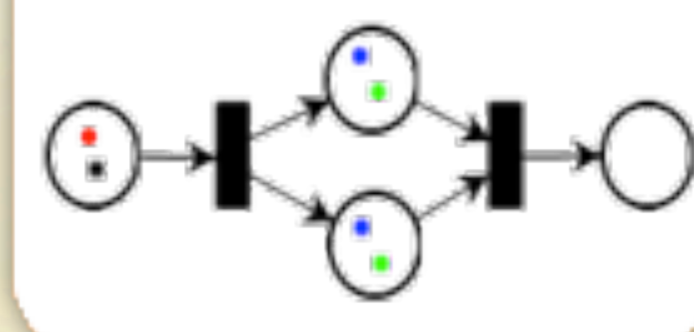
Electrical Circuits



Quantum Processes



Petri Nets

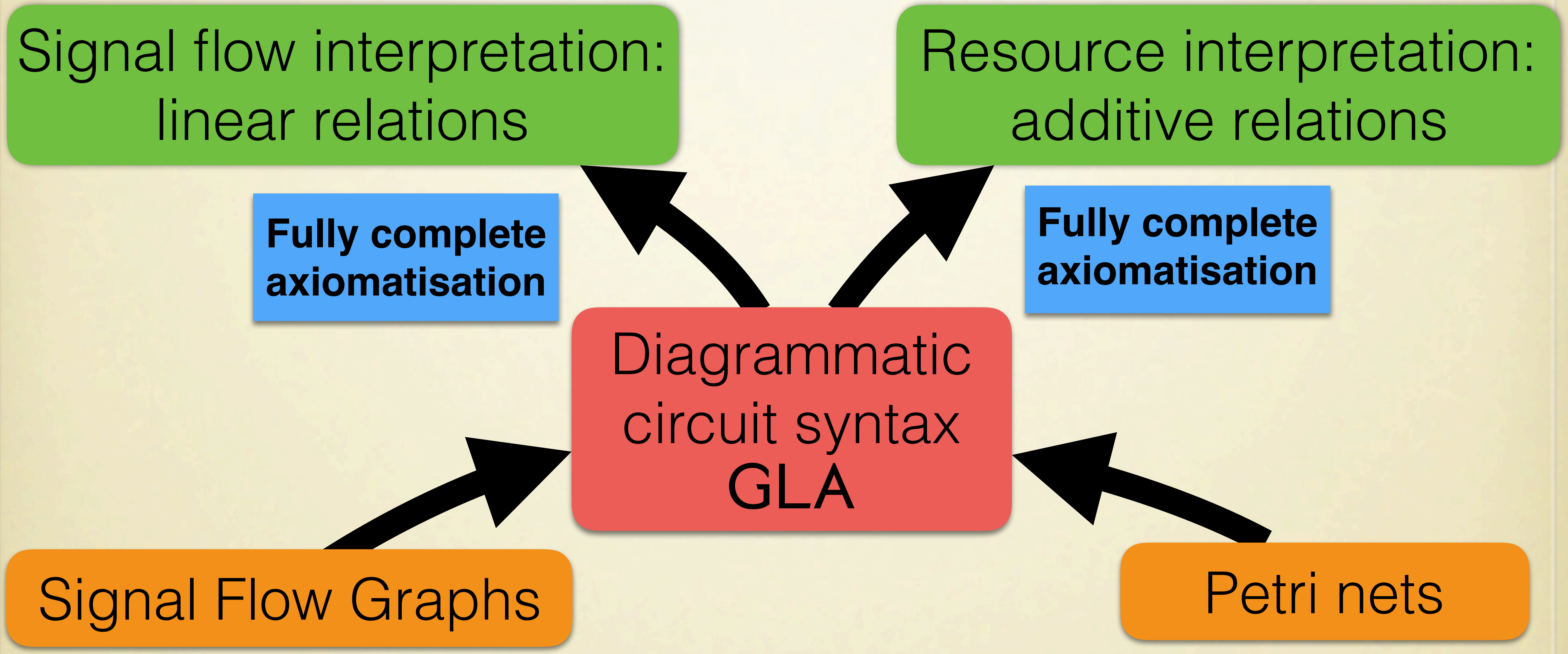


Compositional Modelling

There is an emerging, multi-disciplinary field aiming at studying graphical models of computation **compositionally**, inspired by the **algebraic methods** of programming language semantics.

Diagrams are first-class citizens of the theory. The appropriate algebraic setting is **monoidal** (and not **cartesian**) categories.

In this talk



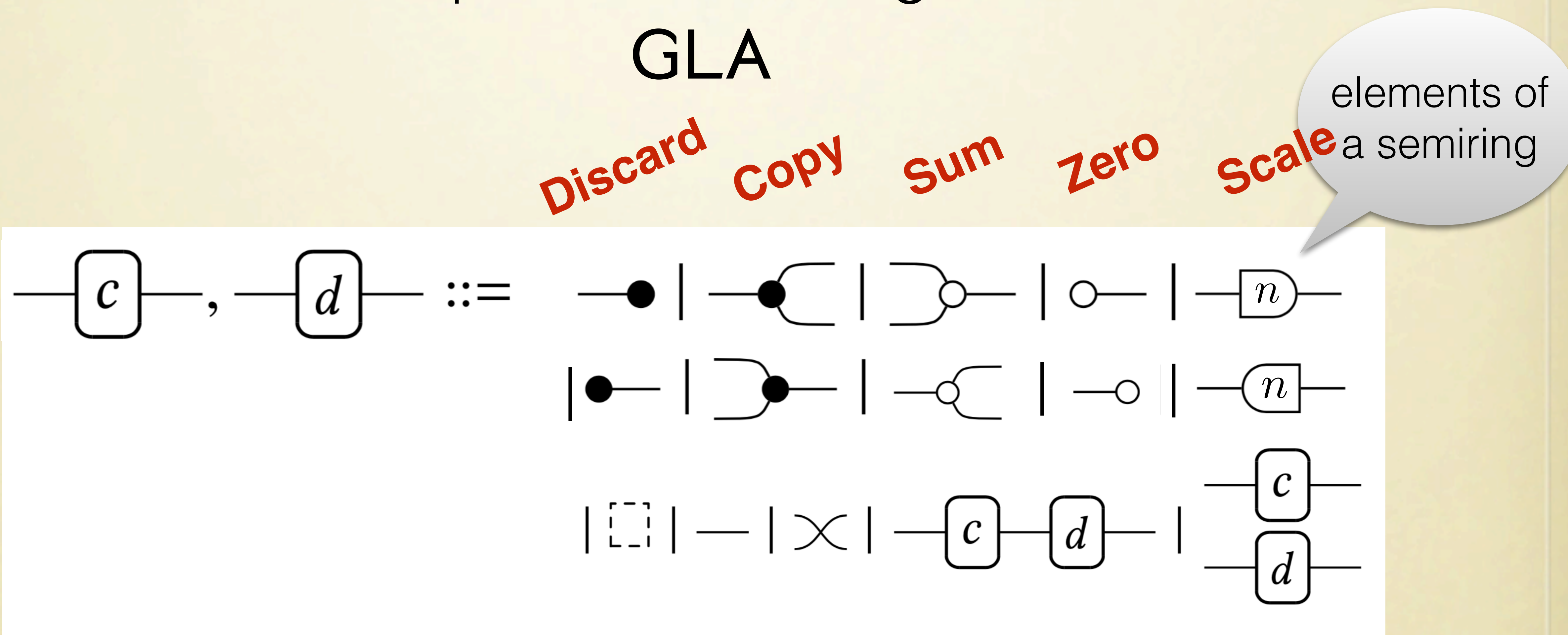
Take home message:
seemingly diverse computational models can be studied within **the same** algebraic framework.

The core language

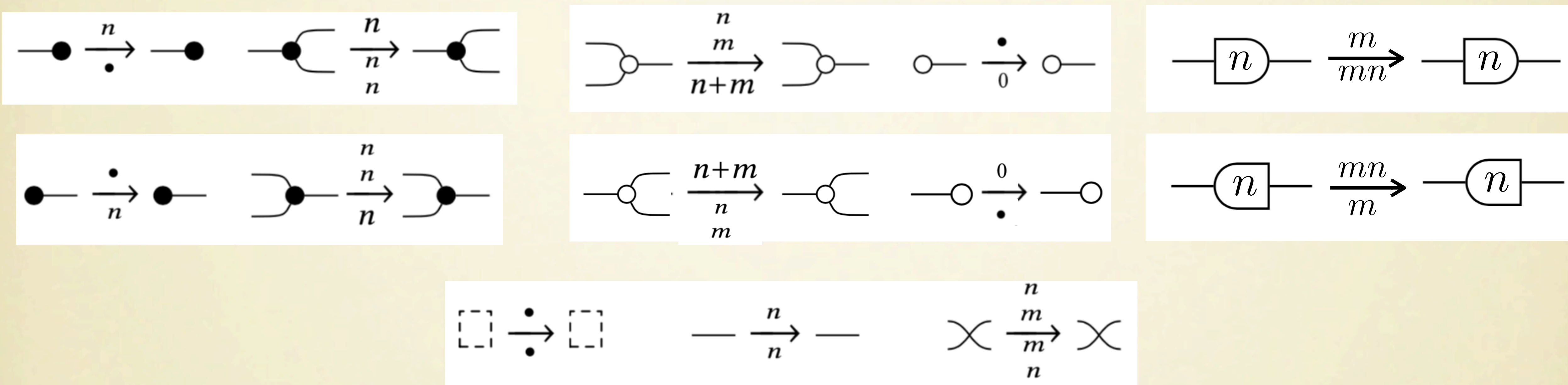
A simple diagrammatic syntax

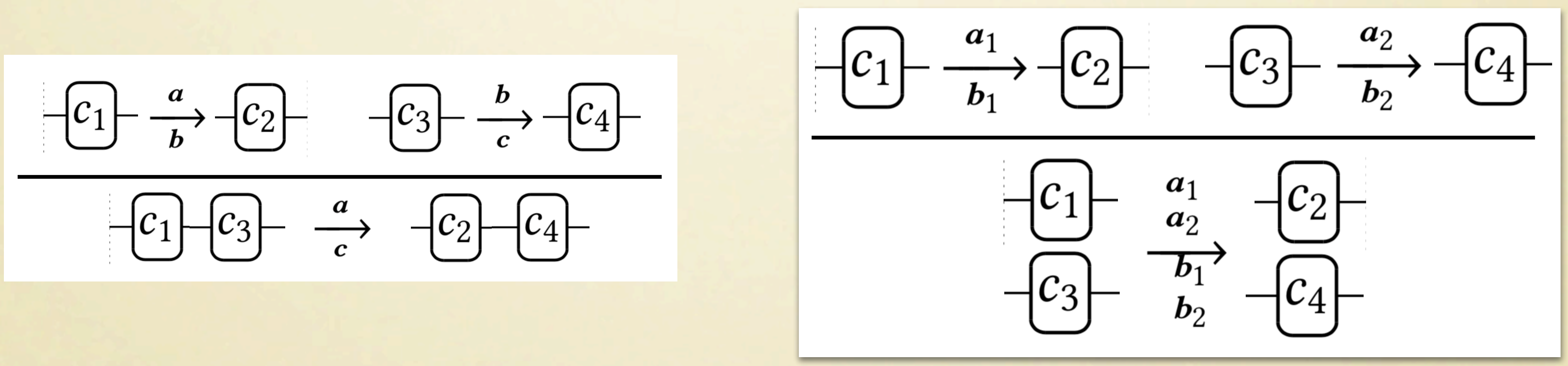
Graphical Linear Algebra

GLA



Circuit Diagrams Behaviour

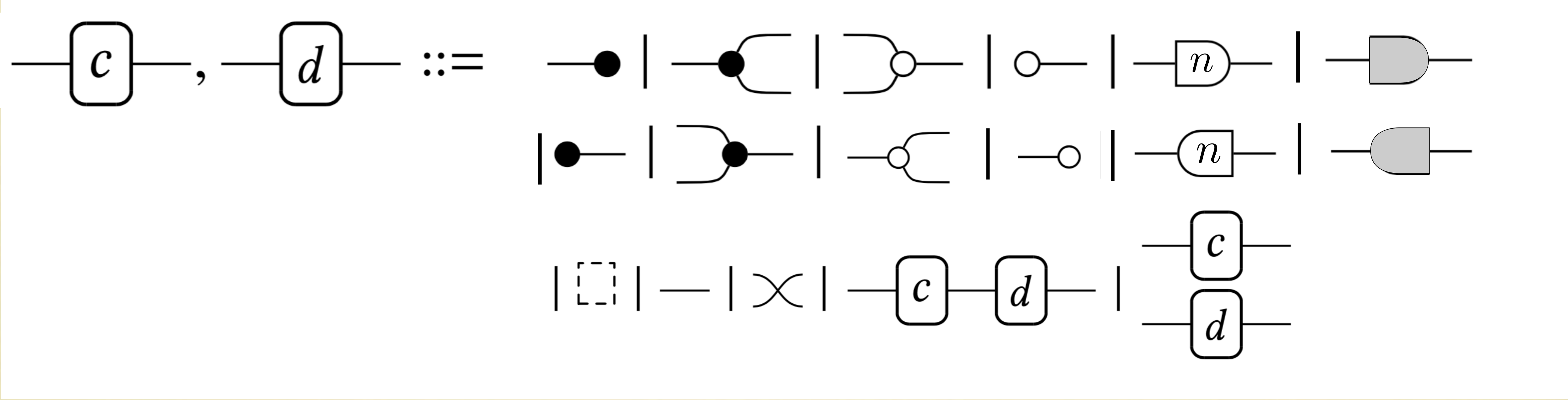




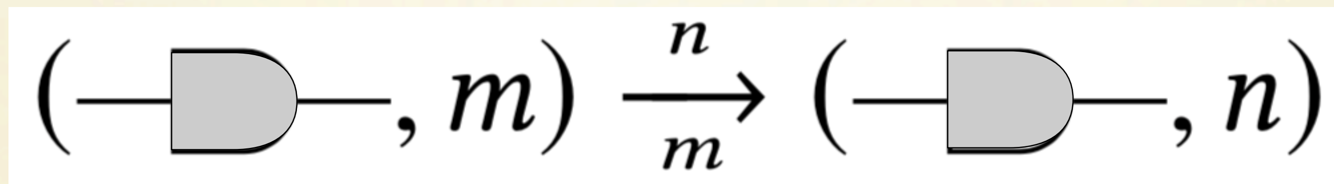
$$[\![c]\!] := \{(a,b) \mid c \xrightarrow[b]{a} c\}$$

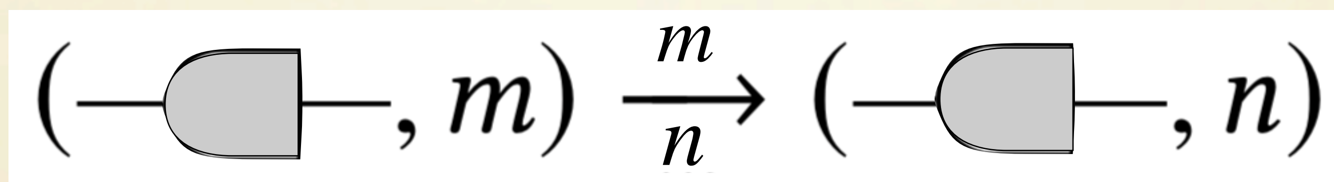
Stateful extension: syntax

GLA$_s$



Stateful extension: behaviour





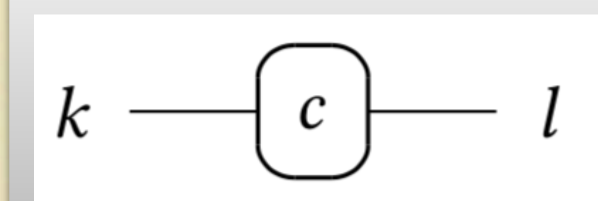
Stateless semantics can be inductively extended to a stateful one:

$$[\![c]\!] := \{(s_1, a, s_2, b) \mid (c, s_1) \xrightarrow[b]{a} (c, s_2)\}$$

The register is canonical

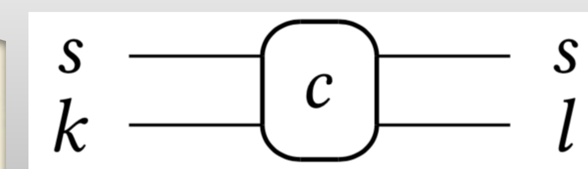
Registers and their semantics are justified by the iso

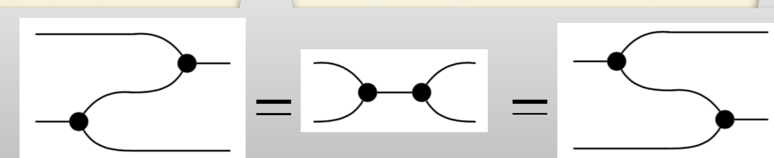
Arrows $k \to l$ are

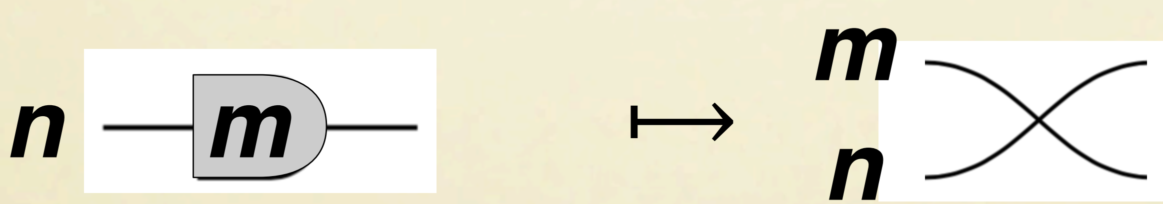


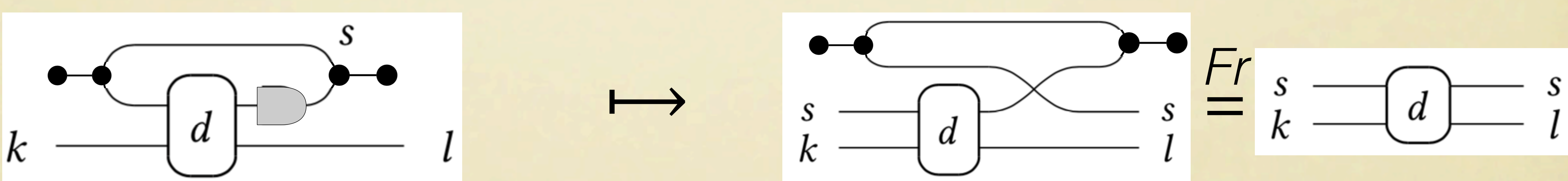
$$(\mathrm{GLA}_s)_{/Fr} \cong \mathrm{St}(\mathrm{GLA}_{/Fr})$$

Arrows $k \to l$ are









What's coming up

The signal flow perspective

(CONCUR'14, POPL'15, *Inf. and Comp.* '17)

Linear interpretation

$$[\![c]\!] := \{(\boldsymbol{a}, \boldsymbol{b}) \mid c \xrightarrow[\boldsymbol{b}]{\boldsymbol{a}} c\}$$

a and ***b*** are now vectors over $\mathbb{R}$.

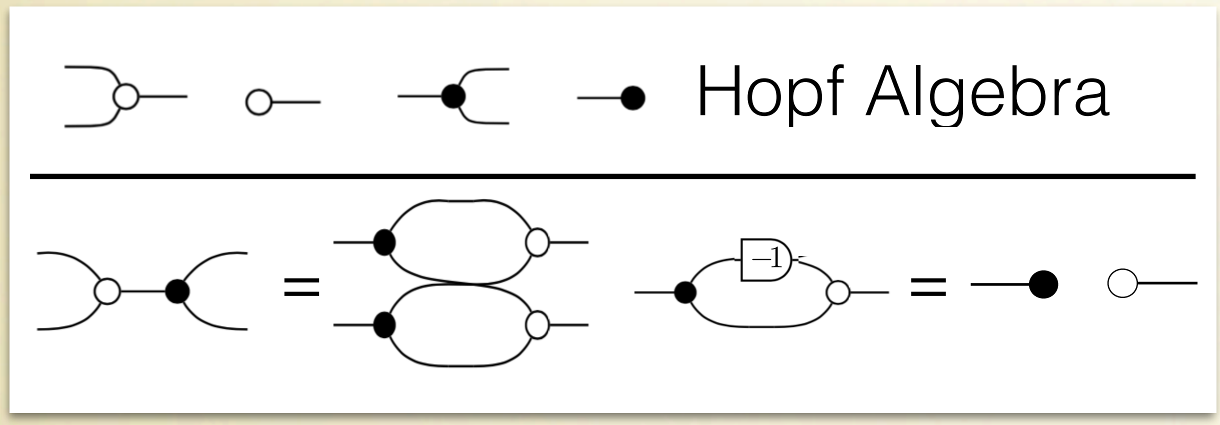
$[\![c\colon k \to l\,]\!]$ is always a ***linear relation*** between $\mathbb{R}^k$ and $\mathbb{R}^l$ (= a subspace of $\mathbb{R}^k \times \mathbb{R}^l$).

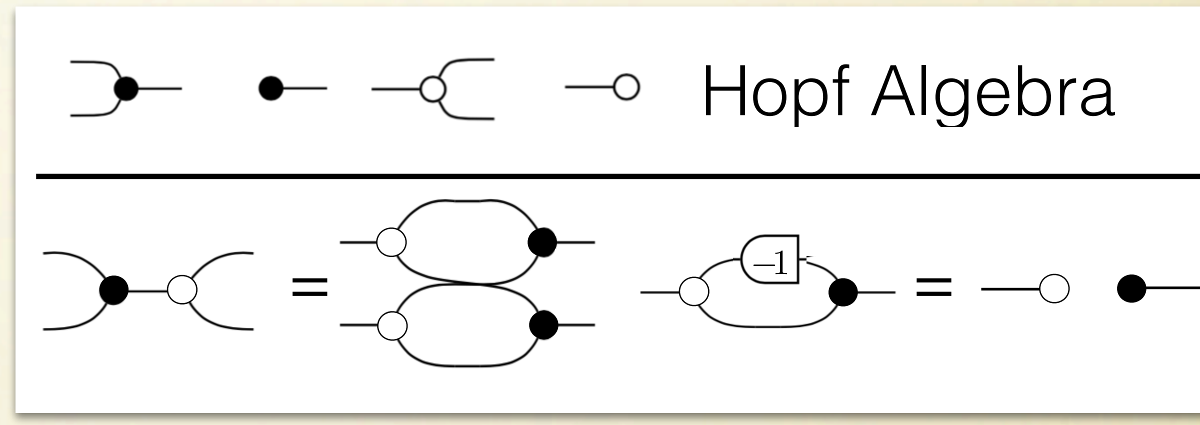
Proposition

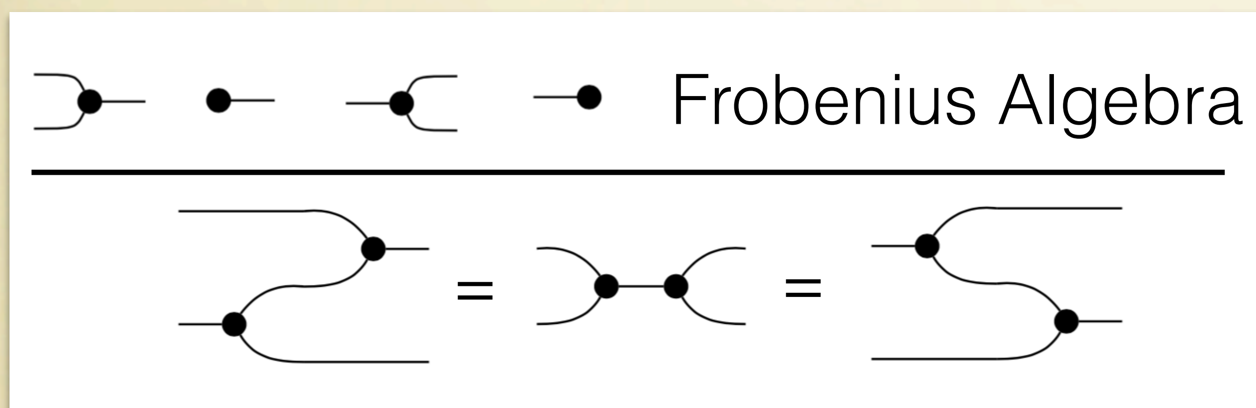
Finite-dimensional linear relations form a category $\mathbf{LinRel}_{\mathbb{R}}$

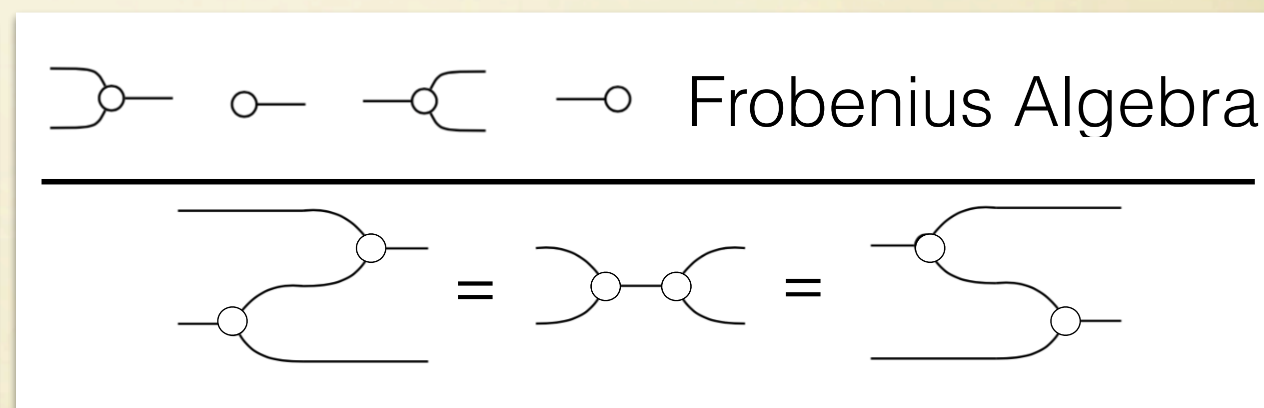
Equational Theory

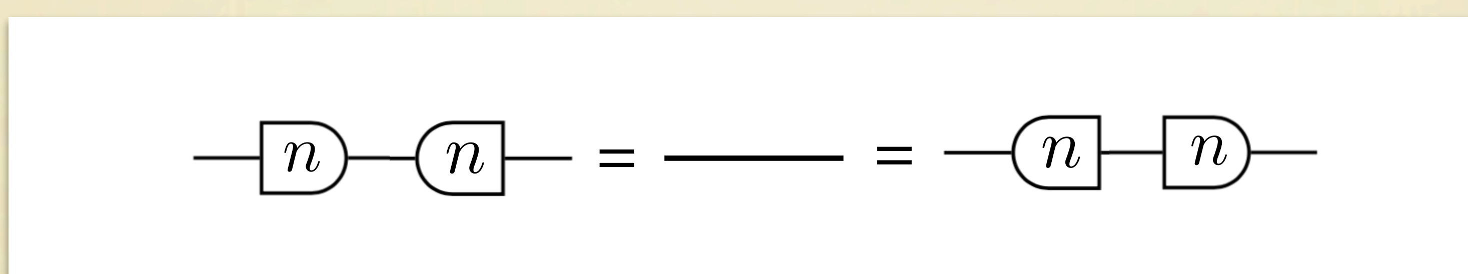
𝕀ℍ: Interacting Hopf Algebras











$n \neq 0$

Completeness

Theorem $\mathsf{GLA}_{/\mathbb{IH}} \cong \mathsf{LinRel}_{\mathbb{R}}$

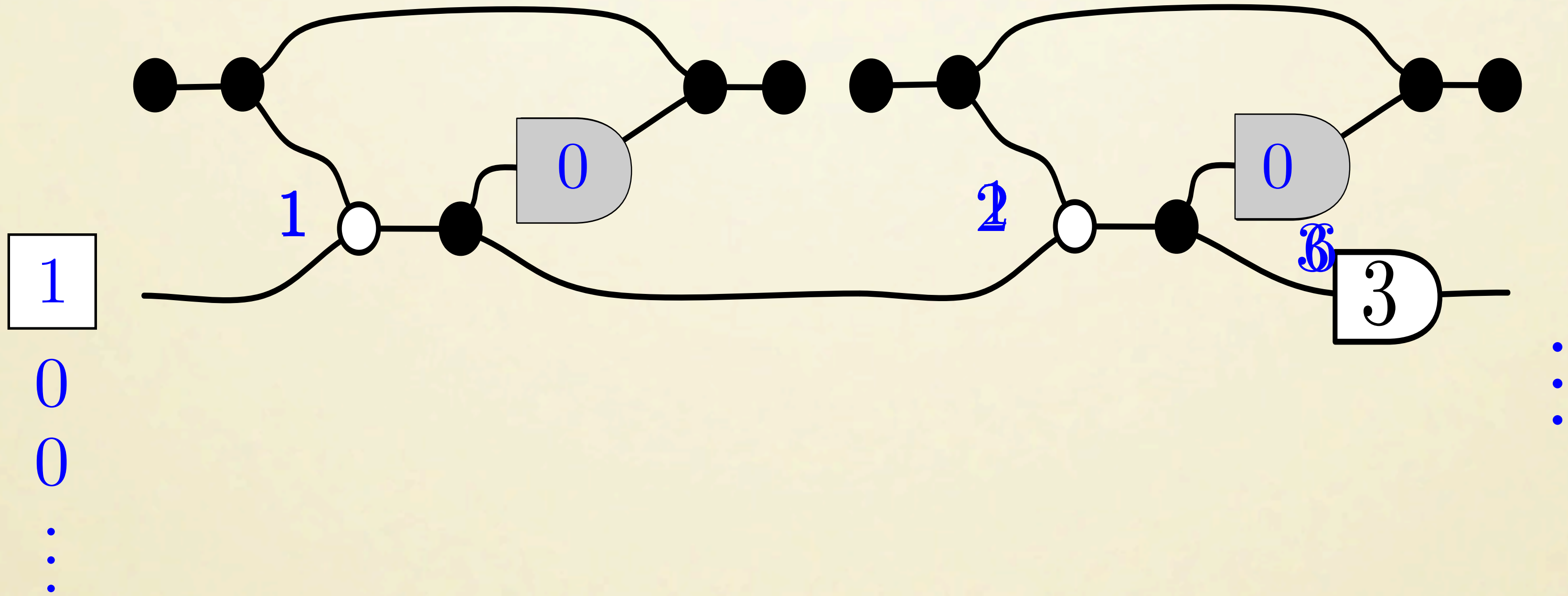
Corollary

$$[\![c]\!] = [\![d]\!] \iff c \stackrel{\mathbb{IH}}{=} d$$

Corollary

$$(\mathsf{GLA}_s)_{/\mathbb{IH}} \cong \mathsf{St}(\mathsf{LinRel}_{\mathbb{R}})$$

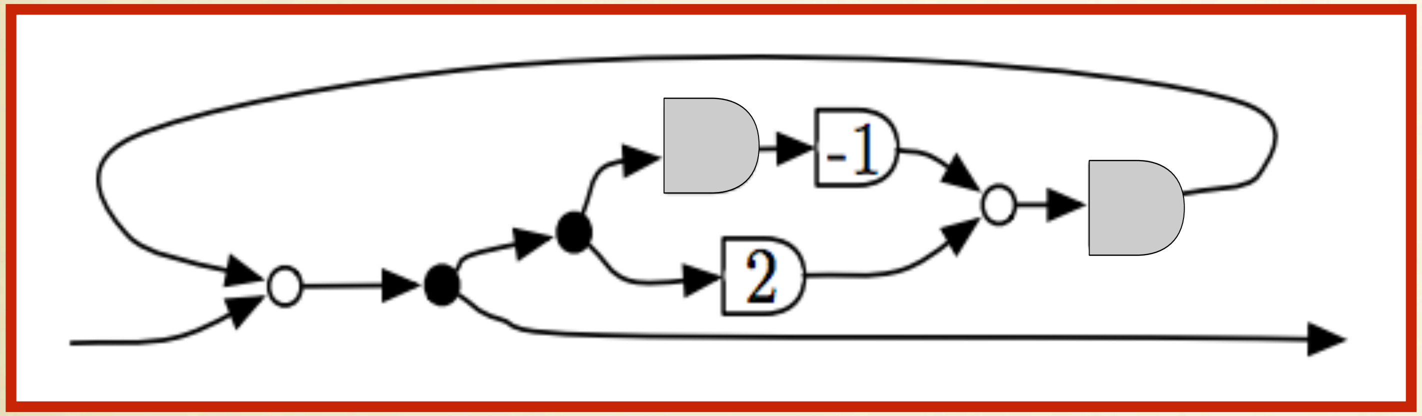
Stateful Example



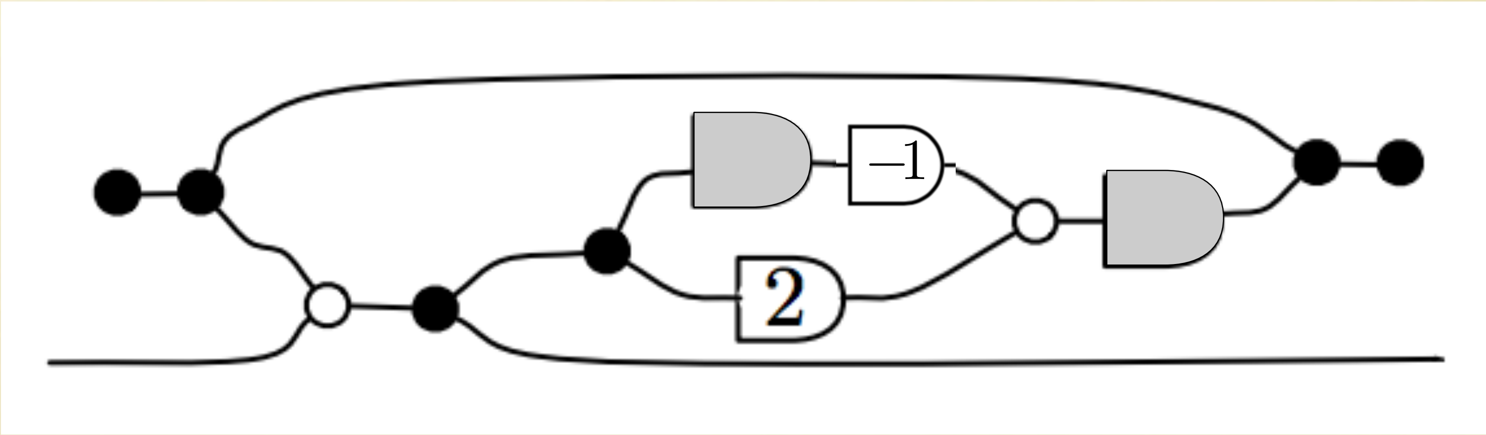
$$(c,0,0) \xrightarrow[3]{1} (c,1,1) \xrightarrow[6]{0} (c,1,2) \xrightarrow[9]{0} (c,1,3) \xrightarrow[12]{0} \ldots$$

Signal Flow Graphs

In fact, the class of **signal flow graphs** embeds in GLA_s



$\longmapsto$



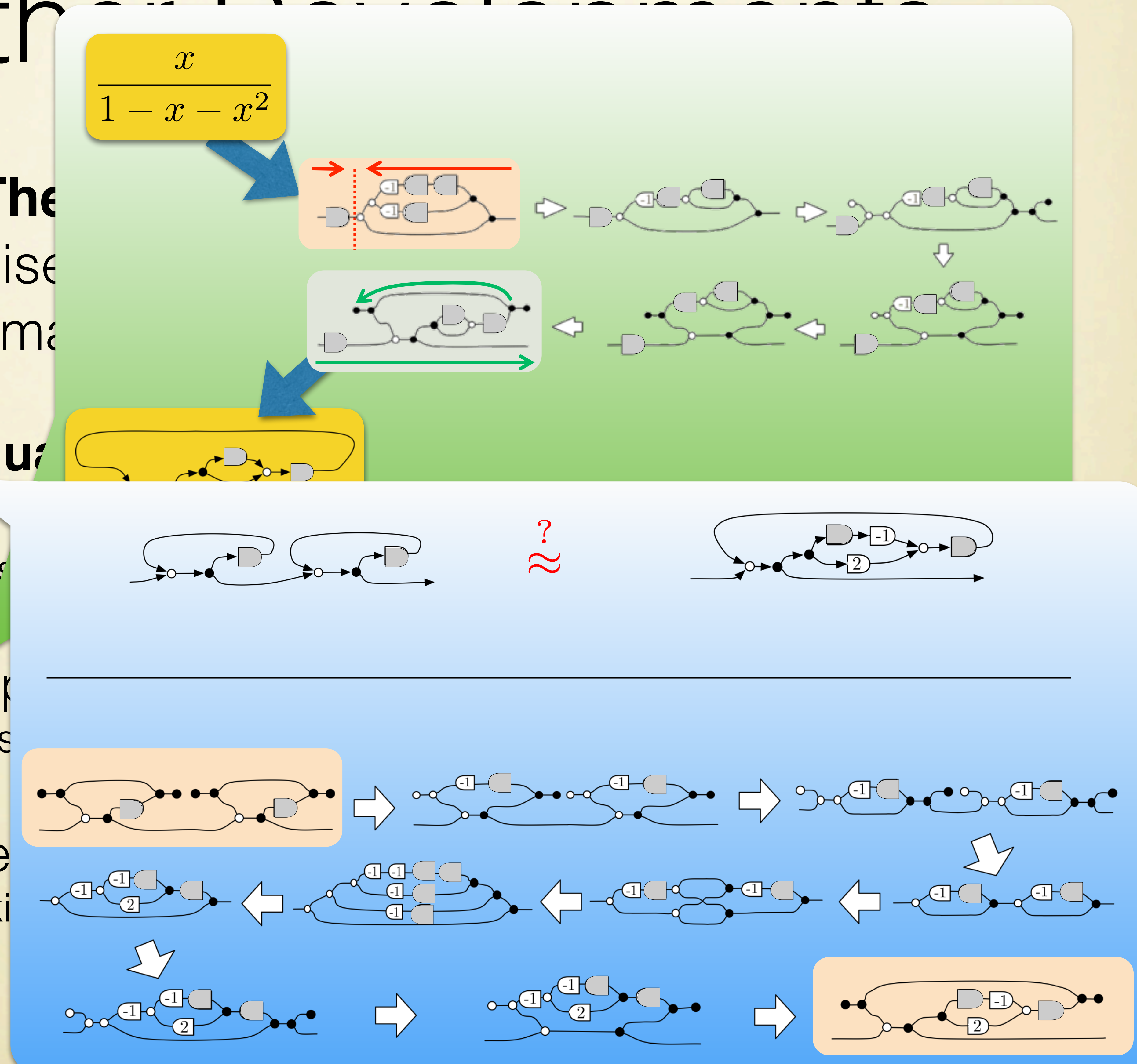
Furth

- ``**Kleene's The**
  identify precise
  semantic doma

- **Graphical Equa**
  Systems.
  (Bonchi, Sobocins

- **Realisability** p
  (Bonchi, Sobocins

- Syntactic crite
  (Fong, Sobocinski

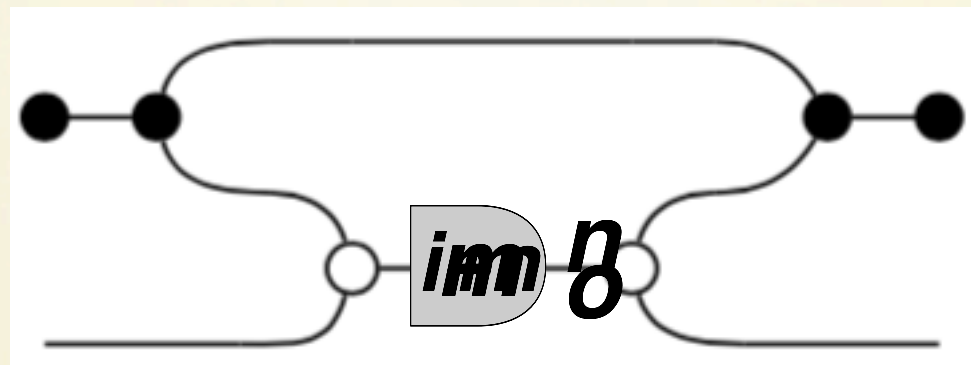


The resource perspective

(POPL'19, LICS'19)

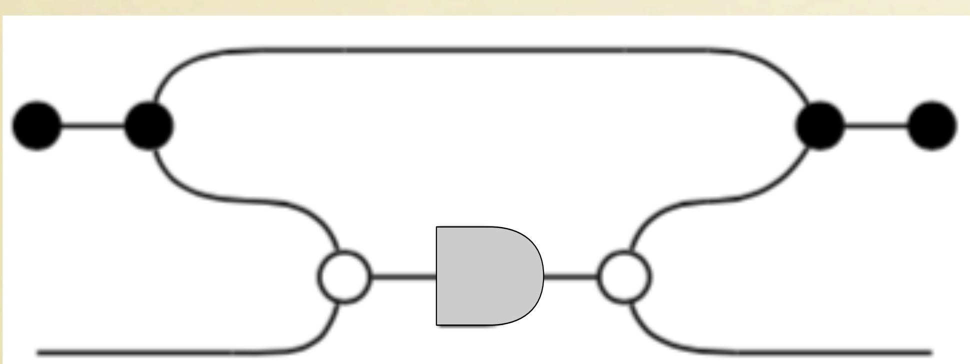
Motivating Example

i



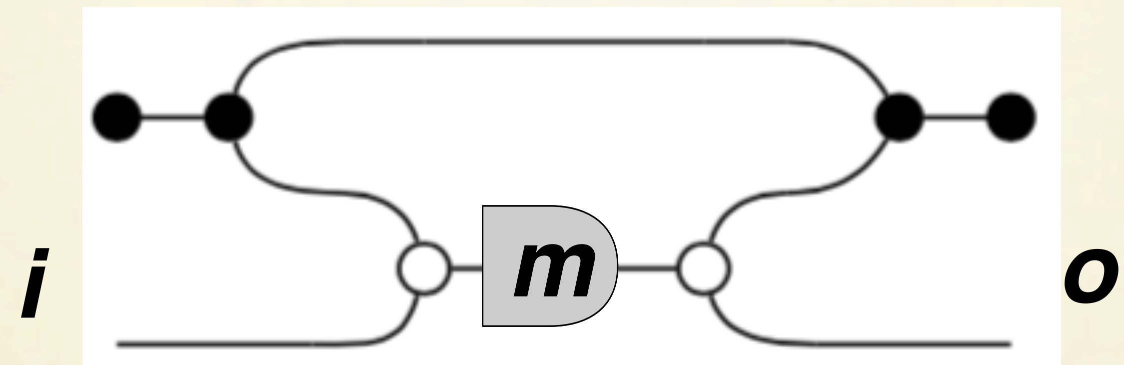
m=n+o

Over $\mathbb{R}$, any ***i***, ***o*** and ***m*** will be in this relation.

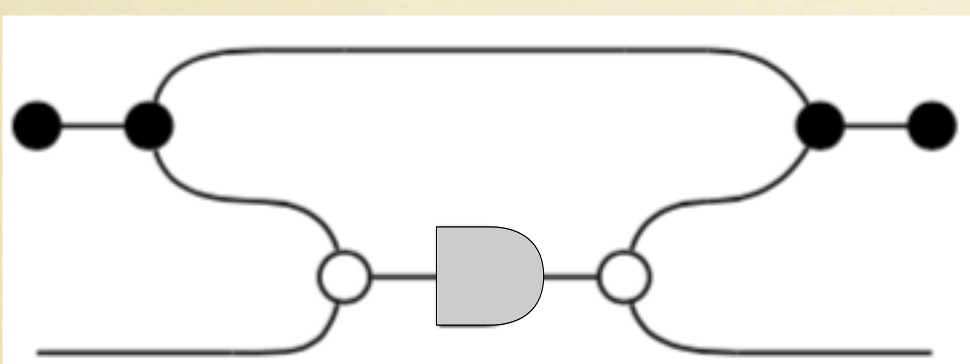


$\stackrel{\mathbb{IH}}{=}$

Motivating Example

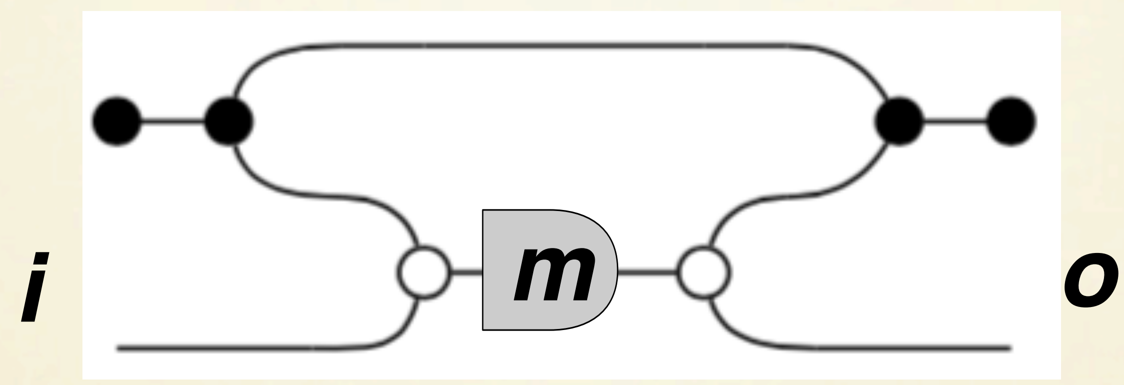


Over $\mathbb{N}$, ***i***, ***o*** and ***m*** are in the relation only if $\boldsymbol{m} \geq \boldsymbol{o}$.

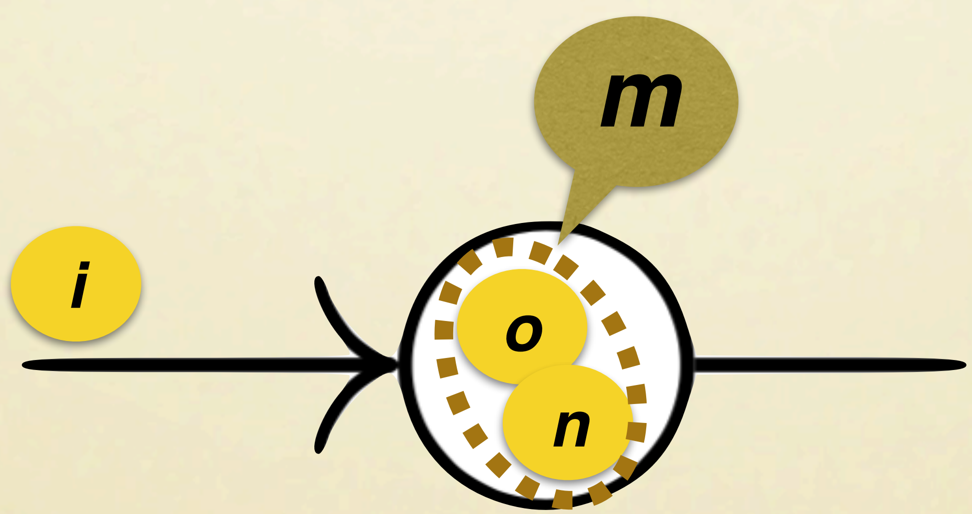


$\neq$

Motivating Example



In fact, over $\mathbb{N}$, this circuit behaves as the **place of a Petri net.**



Additive Relations

$$[\![c]\!] := \{(\boldsymbol{a}, \boldsymbol{b}) \mid c \xrightarrow[\boldsymbol{b}]{\boldsymbol{a}} c\}$$

a and ***b*** are now vectors over $\mathbb{N}$.

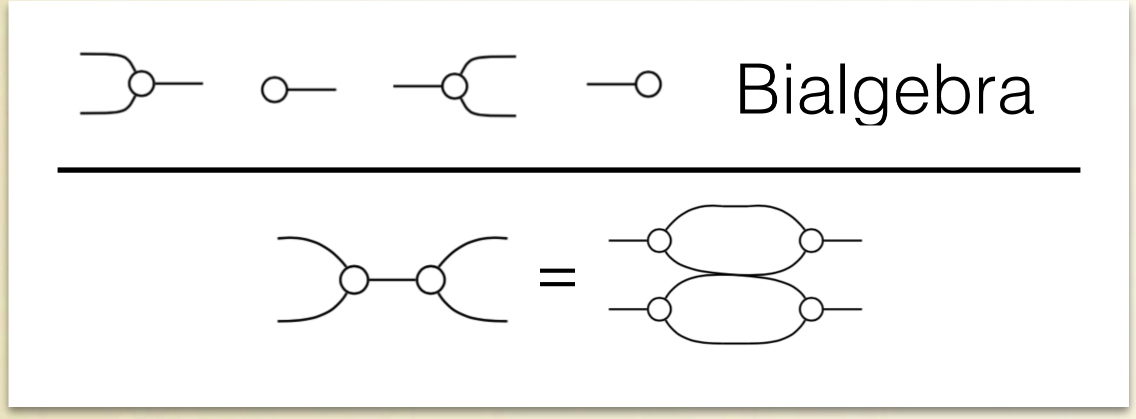
$[\![c : k \to l\,]\!]$ is always an ***additive relation*** between $\mathbb{N}^k$ and $\mathbb{N}^l$
(= a subset of $\mathbb{N}^k \times \mathbb{N}^l$ including (**0**,**0**) and closed under addition)

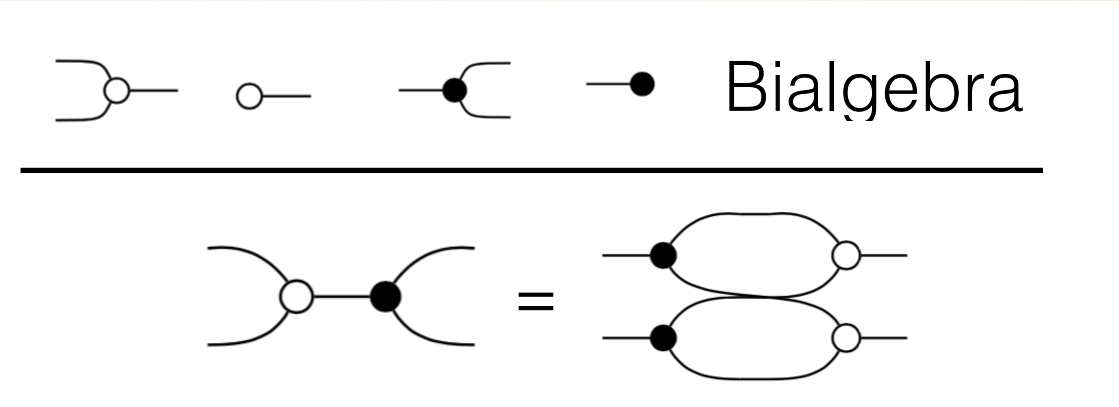
Proposition (non-trivial!)

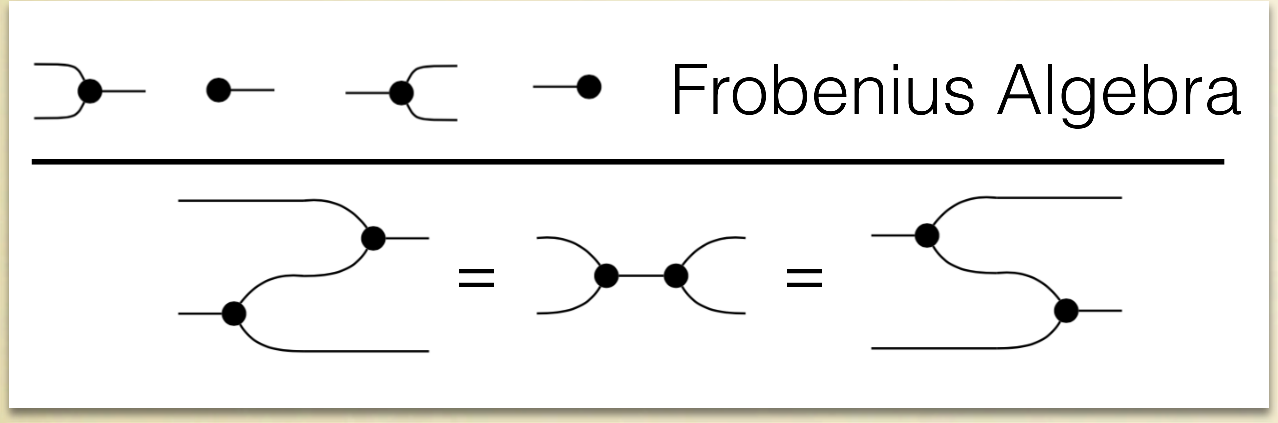
Finitely-generated additive relations form a category $\mathbf{AddRel}_{\mathbb{N}}$

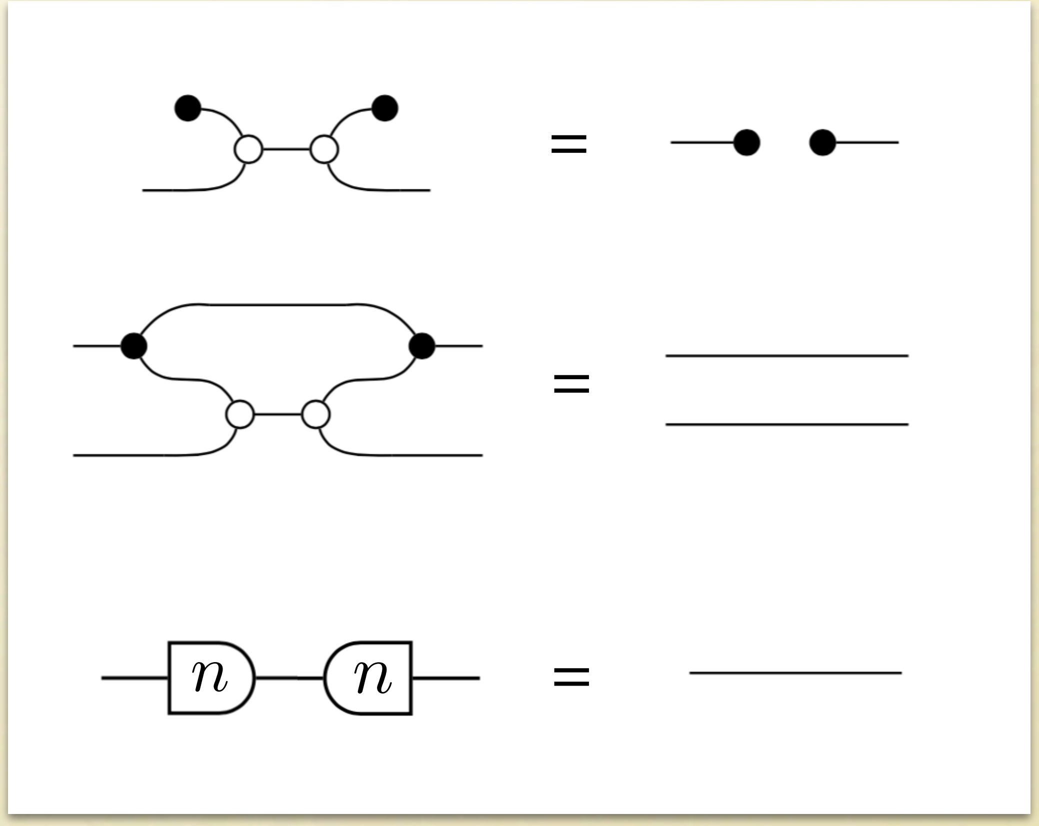
Equational Theory

𝔸ℝ: Algebra of Resources



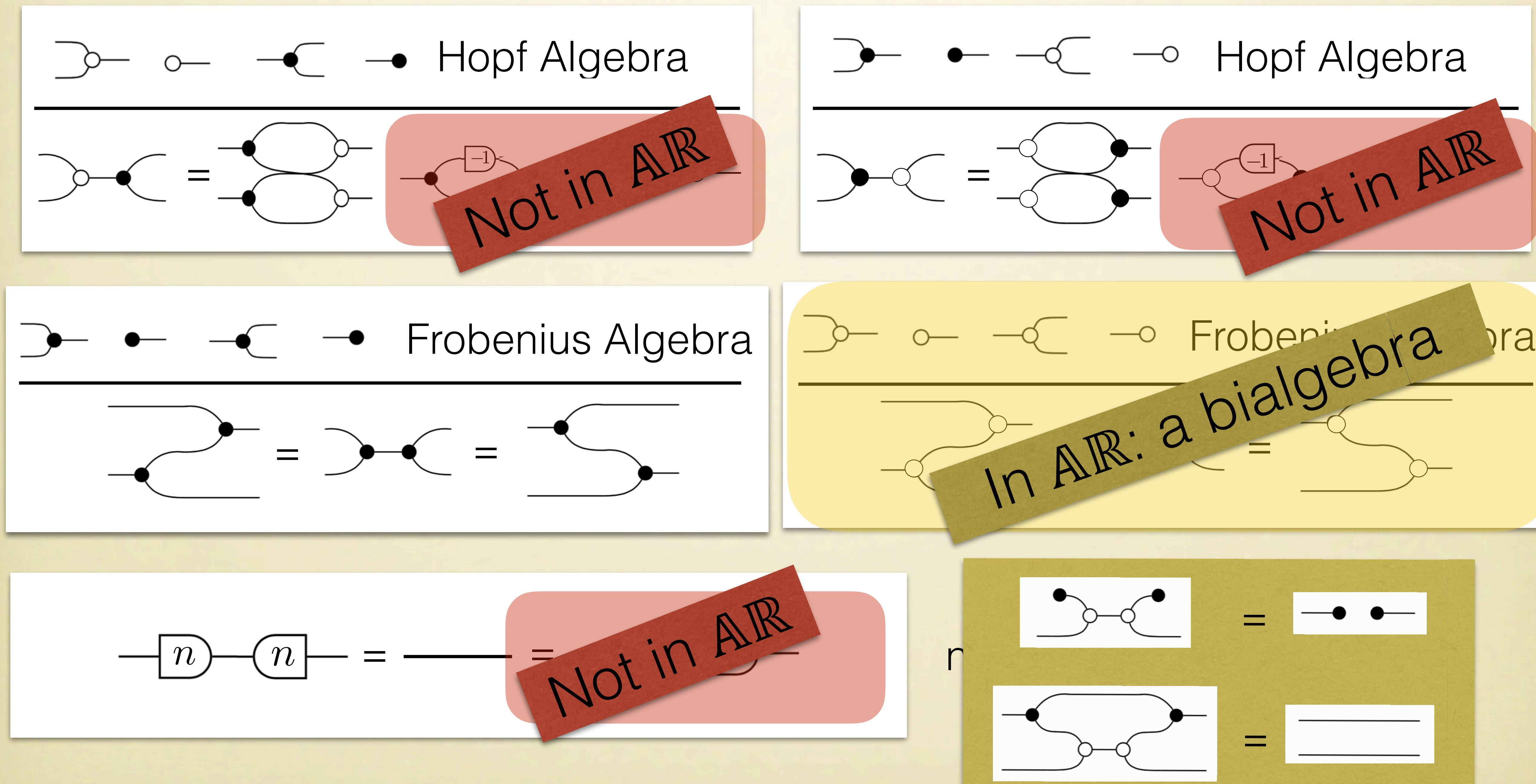






$n \neq 0$

$\mathbb{AR}$ vs $\mathbb{IH}$



Completeness

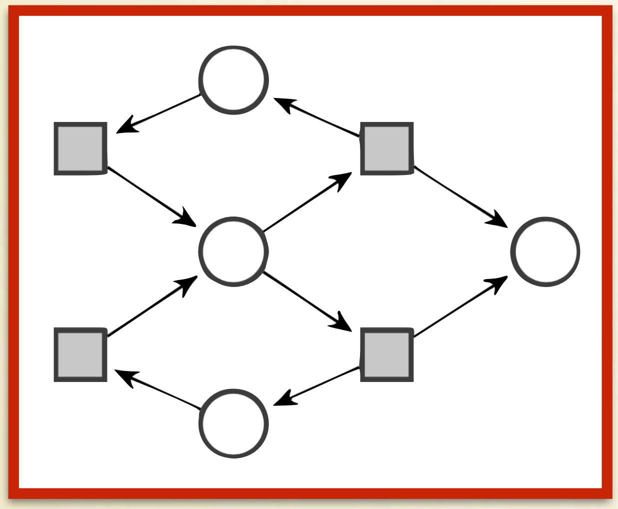
Theorem $\mathsf{GLA}_{/\mathbb{AR}} \cong \mathsf{AddRel}_{\mathbb{N}}$

$$[\![c]\!] = [\![d]\!] \iff c \overset{\mathbb{AR}}{=} d$$

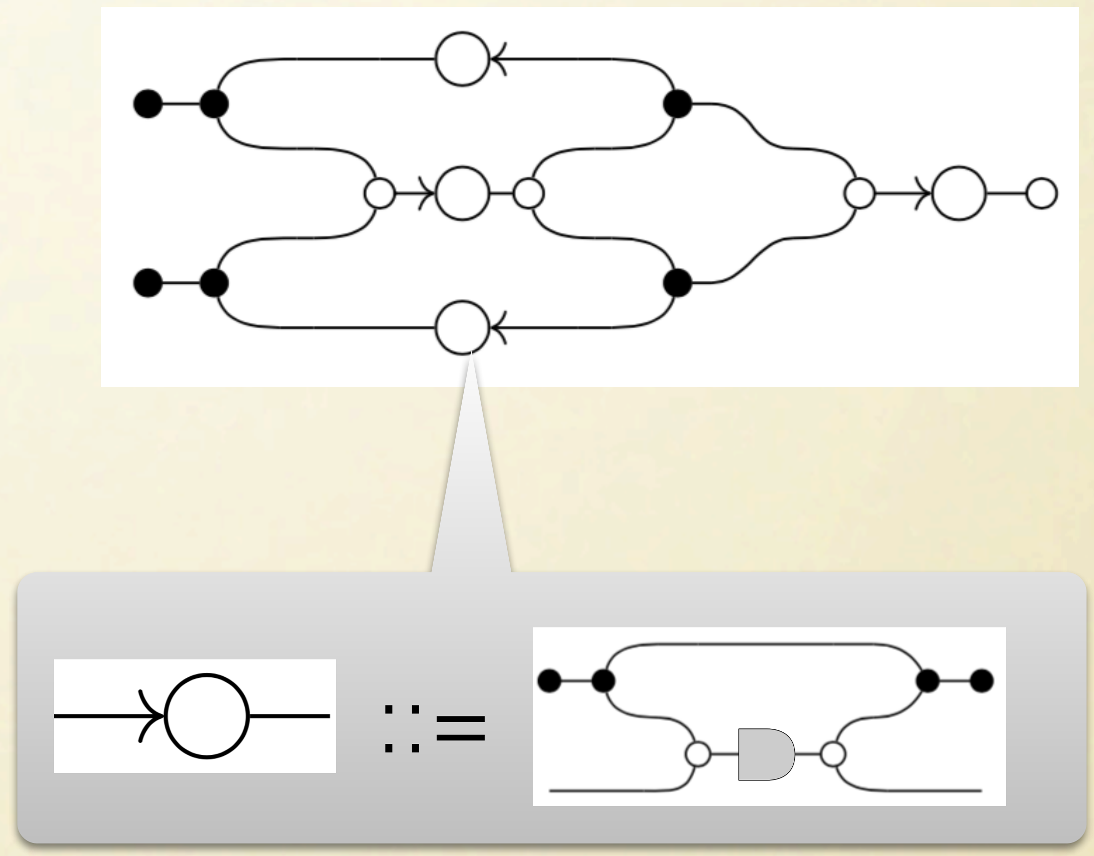
Corollary

$$(\mathsf{GLA}_s)_{/\mathbb{AR}} \cong \mathsf{St}(\mathsf{AddRel}_{\mathbb{N}})$$

Embedding Petri Nets

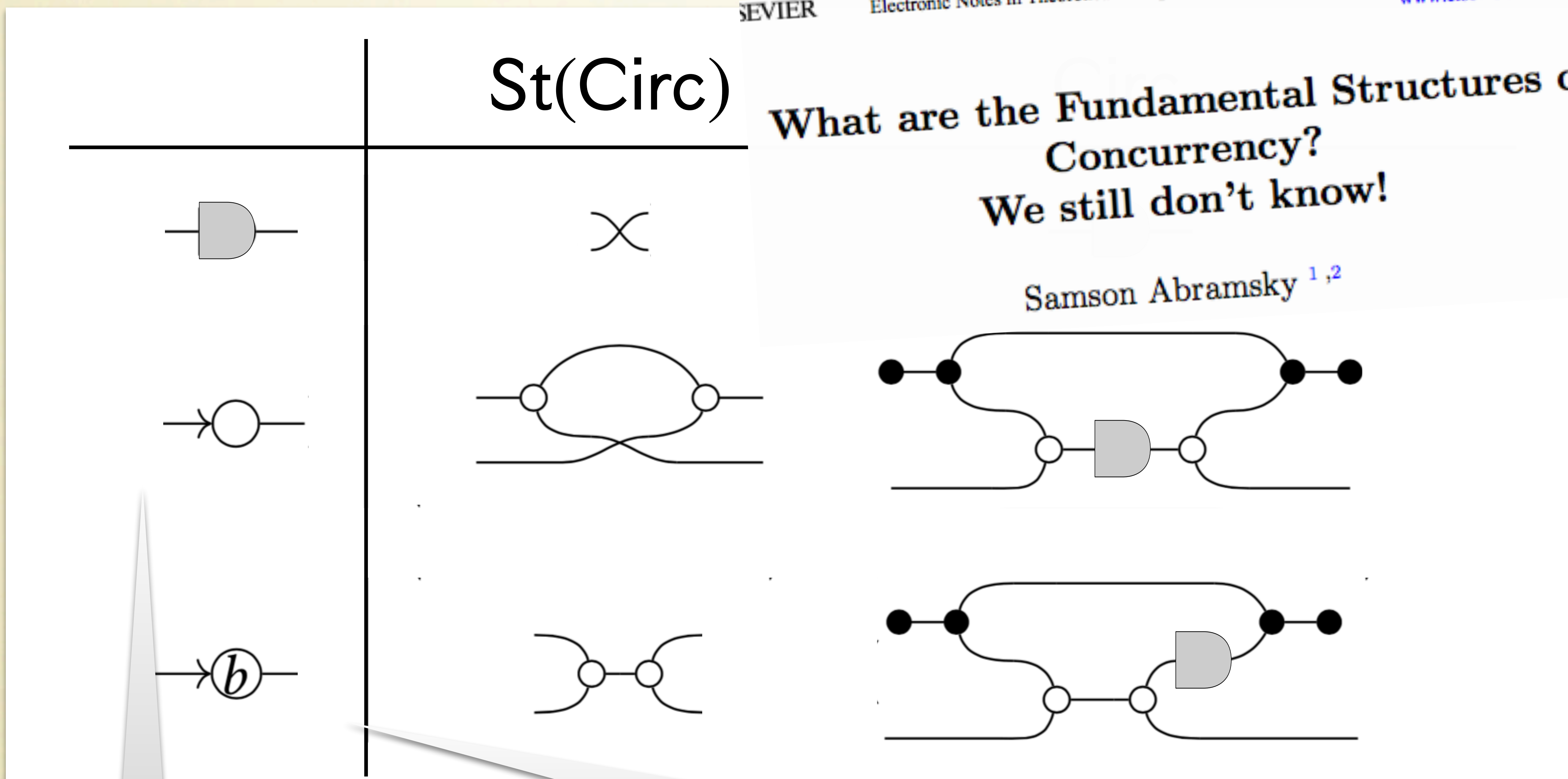


$\longmapsto$



We can thus use $\mathbb{AR}$ for equational reasoning about Petri Nets.

Classifying State Semantics



SEVIER Electronic Notes in Theoretical Computer Science 162 (2006) 37–41 www.elsevier.com/locate/

What are the Fundamental Structures of Concurrency? We still don't know!

Samson Abramsky [1,2]

Firing semantics

$$\frac{o \leq m}{(\rightarrow\!\bigcirc\!-, m) \xrightarrow[o]{i} (\rightarrow\!\bigcirc\!-, m - o + i)}$$

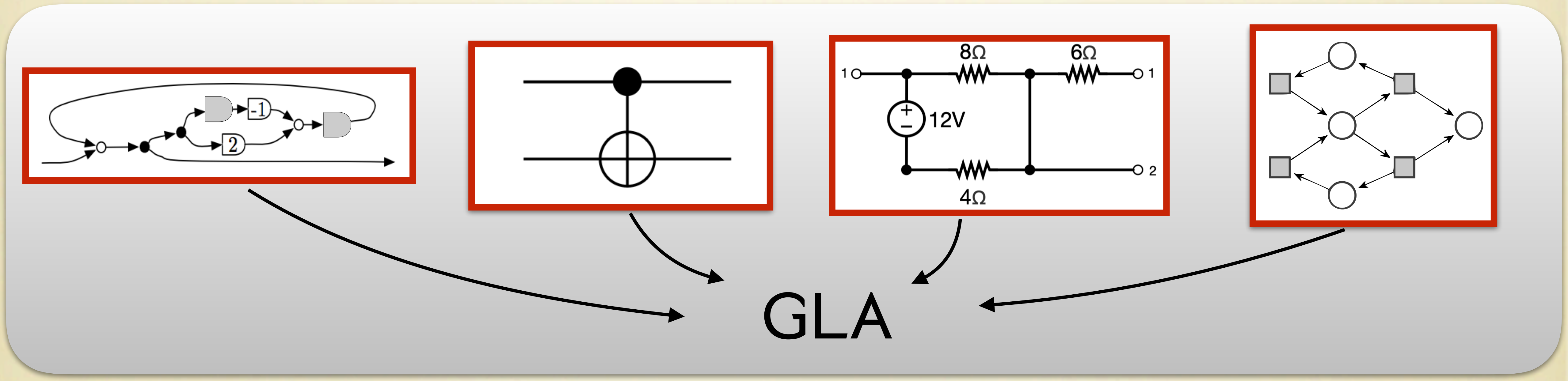
Banking semantics

$$\frac{m + i = m' + o}{(\rightarrow\!\textcircled{b}\!-, m) \xrightarrow[o]{i} (\rightarrow\!\textcircled{b}\!-, m')}$$

Conclusions

Vision

We propose GLA as an **assembly language** for diverse families of component-based systems.



Major research thread: cover **probabilistic** models (Bayesian reasoning, machine learning, etc.).

Bibliography

On the Mathematics Behind Graphical Linear Algebra

- Zanasi - *Interacting Hopf Algebras: the Theory of Linear Systems*, PhD thesis, 2015
- Bonchi, Sobocinski, Zanasi - *Interacting Hopf Algebras*, Journal of Pure and Applied Algebra, 2017.
- www.graphicallinearalgebra.net

On the Signal Flow Perspective

- Bonchi, Sobocinski, Zanasi - *Full Abstraction for Signal Flow Graphs*, POPL, 2015.
- Bonchi, Sobocinski, Zanasi - *The Calculus of Signal Flow Diagrams I: Linear Relations on Streams*, Information and Computation, 2017.

On the Resource Perspective

- Bonchi, Holland, Piedeleu, Sobocinski, Zanasi - *Diagrammatic Algebra: From Linear to Concurrent Systems*, POPL, 2019.
- Bonchi, Piedeleu, Sobocinski, Zanasi - *Graphical Affine Algebra*, LICS, 2019.