A Coalgebraic View on Reachability

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22nd March 2019

To the memory of Vêra Trnková

Abstract

Coalgebras for an endofunctor provide a category-theoretic framework for modeling a wide range of state-based systems of various types. We provide an iterative construction of the reachable part of a given pointed coalgebra that is inspired by and resembles the standard breadth-first search procedure to compute the reachable part of a graph. We also study coalgebras in Kleisli categories: for a functor extending a functor on the base category, we show that the reachable part of a given pointed coalgebra can be computed in that base category.

1 Introduction

Coalgebras provide a convenient category theoretic framework in which to model state-based systems and automata whose transition type is described by an endofunctor. For example, classical deterministic and non-deterministic automata, labelled transition systems as well as their weighted and probabilistic variants arise as instances of coalgebras.

A key notion in the theories of state-based systems of various types is reachability, i.e. the construction of a subsystem of a given system containing precisely those states that can be reached from (a set of) initial states along a path in the transition graph of the system. For example, in automata theory, computing the reachable part of a given deterministic automaton is the first step in every minimization procedure. It is well-known that reachability has a simple formulation

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on the level of coalgebras. In fact, a pointed coalgebra, i.e. one with a given initial state, is called reachable if it does not contain any proper subcoalgebra containing the initial state [4]. Moreover, for a functor preserving intersections, the reachable part of a given pointed coalgebra is obtained by taking the intersection of all the subcoalgebras containing the initial state. The purpose of the present paper is a more thorough study of reachable coalgebras and, in particular, a new iterative construction of the reachable part of a given pointed coalgebra.

After recalling some preliminaries in Section 2, we discuss some background material on endofunctors on Set preserving intersections in Section 3 and on the canonical graph of a coalgebra in Set in Section 4.

In Section 5 we present a new iterative construction of the reachable part of a given pointed coalgebra that is inspired by and closely resembles the standard breadth-first search in graphs. Our construction works for coalgebras over every well-powered category $\mathscr{C}$ having coproducts and a factorization system $(\mathscr{E}, \mathscr{M})$, where $\mathscr{M}$ consists of monomorphisms. Moreover, the coalgebraic type functor $F\colon \mathscr{C} \to \mathscr{C}$ is assumed to have *least bounds*, a notion previously introduced by Block [10]. Extending a result by Gumm [13] for set functors, we prove in Proposition 5.9 that a functor has least bounds if and only if it preserves intersections. Moreover, this is equivalent to the existence of a left-adjoint to the operator $\bigcirc_f\colon \mathsf{Sub}(Y) \to \mathsf{Sub}(X)$ for every $f\colon X \to FY$, which assigns to every subobject m of Y the pullback of Fm along f. Note that, for a coalgebra $c\colon C \to FC$, this operator is Jacobs' "next time" operator [15]. In our iterative construction of the reachable part we use its left-adjoint $\ominus_c$ on $\mathsf{Sub}(C)$, which corresponds to the "previous time" operator of classical linear temporal logic [19]. In fact, we consider a coalgebra $c\colon C \to FC$ together with an *I-pointing*, i.e. a morphism $i_C\colon I \to C$, where I is some object, and we prove in our main result Theorem 5.20 that the reachable part of the given I-pointed coalgebra is given by the union of all $\ominus^k(m_0)$, where m_0 is given by $(\mathscr{E}, \mathscr{M})$-factorizing the given I-pointing i_C. Moreover, we prove that, whenever F preserves inverse images, the reachable part is a coreflection of (C, c, i_C) into the category of I-pointed reachable F-coalgebras (Theorem 5.23). We also show that for an I-pointed coalgebra in Set the above iterative construction of the reachable part can be performed as a standard breadth-first search on the canonical graph (Corollary 5.26).

Finally, we study in Section 6 coalgebras for a functor $\bar{F}$ on a Kleisli category over $\mathscr{C}$, which is an extension of an endofunctor F on $\mathscr{C}$. Here we show that the reachable part of a given I-pointed $\bar{F}$-coalgebra can be constructed as the reachable part of a related coalgebra in $\mathscr{C}$.

Dedication. We would like to dedicate this paper to the memory of Vêra Trnková. Her research, especially her foundational results of the late 1960s and early 1970s on set functors, are still continuing to have considerable impact, in particular for the theory of coalgebras. In addition, her work on Kleisli categories and lifting functors to categories of relations is of basic importance for work on coalgebraic logic. We make use of Trnková's careful research on properties of set functors in Section 3.

Related work. Our results are based on the notion of reachable coalgebras introduced by Adámek et al. [4]. Our construction of the reachable part appears in work by Wißmann, Dubut, Katsumata, and Hasuo [27] (see Lemma A.5 of the full version), where it is used as an auxiliary construction in order to give a characterization of the reachability of a coalgebras in terms of paths [27, Section 3.5]. However, that work does not connect the construction with the "previous time" operator.

The "previous time" operator is also studied by Barlocco, Kupke, and Rot [8]. They work with a complete and well-powered category $\mathscr{C}$, and, like us, they show that the reachable part of a given pointed coalgebra can be obtained by an iterative construction using the "previous time" operator. Their results were obtained independently and almost at the same time as ours.

2 Pointed and Reachable Coalgebras

In this section we recall some preliminaries on pointed and reachable coalgebras for an endofunctor. A coalgebra for an endofunctor $F\colon \mathscr{C} \to \mathscr{C}$ (or F-coalgebra, for short) is a pair (C, c) where C is an object of $\mathscr{C}$ called the *carrier* of the coalgebra and $c\colon C \to FC$ a morphism called the *structure* of the coalgebra. A *coalgebra homomorphism* $h\colon (C, c) \to (D, d)$ is a morphism $h\colon C \to D$ of $\mathscr{C}$ that commutes with the structures on C and D, i.e. the following square commutes:

$$\begin{array}{ccc} C & \xrightarrow{c} & FC \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ D & \xrightarrow{d} & FD \end{array}$$

Definition 2.1. Given an endofunctor $F\colon \mathscr{C} \to \mathscr{C}$ and an object I of $\mathscr{C}$, an *I-pointed F-coalgebra* is a triple (C, c, i_C) where (C, c) is an F-coalgebra and $i_C\colon I \to C$ a morphism of $\mathscr{C}$. A homomorphism of I-pointed coalgebras from (C, c, i_C) to (D, d, i_D) is a coalgebra homomorphism $h\colon (C, c) \to (D, d)$ preserving the pointings, i.e. $h \cdot i_C = i_D$. We denote by

$$\mathsf{Coalg}_I(F)$$

the category of I-pointed F-coalgebra and their homomorphisms.

Example 2.2. Pointed coalgebras allow to capture many kinds of state-based systems categorically. We just recall a couple of examples; for further examples, see e.g. [23].

(1) Deterministic automata are 5-tuples $(S, \Sigma, \delta, s_0, F)$, with a set S of states, an input alphabet Σ, a next-state function $\delta\colon S \times \Sigma \to S$, an initial state $s_0 \in S$ and a set $F \subseteq S$ of final states. Here we fix the input alphabet Σ. Representing the subset F by its characteristic function $f\colon S \to \{0, 1\}$, and currying δ we see

that a deterministic automaton is, equivalently, a pointed coalgebra for $FX = \{0,1\} \times X^\Sigma$ on Set with the pointing $s_0 \colon 1 \to S$ given by the initial state.

(2) Non-deterministic automata are similar to deterministic ones, except that in lieu of a next-state function one has a next-state relation $\delta \subseteq S \times \Sigma \times S$ and a set of initial states $I \subseteq S$. These data can be represented as two functions $i \colon 1 \to \mathcal{P}S$ and $c \colon S \to \mathcal{P}(1+\Sigma\times S)$, where $\mathcal{P}$ denotes the power-set. That means that a non-deterministic automaton is, equivalently, a coalgebra for the functor $FX = 1 + \Sigma \times X$ on the Kleisli category of the monad $\mathcal{P}$, i.e. the category Rel of sets and relations.

(3) Pointed graphs are, equivalently, coalgebras for the power-set functor $\mathcal{P} \colon \mathsf{Set} \to \mathsf{Set}$. Indeed, a pointed coalgebra

$$1 \xrightarrow{v_0} V \xrightarrow{e} \mathcal{P}V$$

consists of a set of vertices V with directed edges given by a binary relation, represented by e, and a distinguished node $v_0 \in S$.

(4) The category of nominal sets provides a framework where freshness of names or resources in systems can be modelled or where systems can store values from infinite data types. We briefly recall the definition of the category Nom of nominal sets (see e.g. Pitts [22]). We fix a countably infinite set $\mathbb{A}$ of *atomic names*. Let $\mathfrak{S}_\mathsf{f}(\mathbb{A})$ denote the group of all finite permutations on $\mathbb{A}$ (which is generated by all transpositions $(a\,b)$ for $a, b \in \mathbb{A}$). Let X be a set with an action of this group, denoted by $\pi \cdot x$ for a finite permutation π and $x \in X$. A subset $A \subseteq \mathbb{A}$ is called a *support* of an element $x \in X$ provided that every permutation $\pi \in \mathfrak{S}_\mathsf{f}(\mathbb{A})$ that fixes all elements of A also fixes x:

$$\forall \pi \in \mathfrak{S}_\mathsf{f}(\mathbb{A}) \colon \pi(a) = a \text{ for all } a \in A \implies \pi \cdot x = x.$$

A *nominal set* is a set with an action of the group $\mathfrak{S}_\mathsf{f}(\mathbb{A})$ such that every element has a finite support. The category Nom is formed by nominal sets and *equivariant maps*, i.e. maps preserving the given group action. Each nominal set X is thus equipped with an equivariant map $\mathsf{supp} \colon X \to \mathcal{P}_\mathsf{f}(\mathbb{A})$ that assigns to each element its least support. For example, the set of terms of the λ-calculus modulo renaming of bound variables is a nominal set, where the least support of a λ-term is the set of its free variables. Variable binding can be modelled by the *binding* functor on Nom. This functor maps a nominal set X to the nominal set $[\mathbb{A}](X) = (\mathbb{A} \times X)/{\sim}$ where $(a, x) \sim (b, y)$ iff $(c\,a) \cdot x = (c\,b) \cdot y$ for any *fresh* c, i.e. $c \notin \mathsf{supp}(x) \cap \mathsf{supp}(y)$. That means that $\sim$ abstracts α-equivalence known from calculi with name binding such as the λ-calculus. In fact, the set of λ-expressions modulo α-equivalence is the initial algebra for the endofunctor $FX = \mathbb{A} + X \times X + [\mathbb{A}]X$ on Nom [12]. Coalgebras for functors on Nom have been studied e.g. in [17, 20, 21].

There are a number of different notions of automata featuring a nominal set of states and which process words over the infinite input alphabet $\mathbb{A}$. One example of a coalgebraic notion of automata are regular nondeterministic nominal automata

(RNNA) [24]; they are precisely the coalgebras for the functor on nominal sets given by

$$FX = 2 \times \mathcal{P}_{\mathsf{ufs}}(\mathbb{A} \times X) \times \mathcal{P}_{\mathsf{ufs}}([\mathbb{A}]X),$$

where $\mathcal{P}_{\mathsf{ufs}}$ is a variant of the finite power-set functor on Nom—it maps a nominal set X to the nominal set of all of its *uniformly supported* subsets S, i.e. S is an equivariant subset of X such that $\bigcup_{x \in S} \mathsf{supp}(x)$ is finite.

Intuitively, in a coalgebra $C \to 2 \times \mathcal{P}_{\mathsf{ufs}}(\mathbb{A} \times C) \times \mathcal{P}_{\mathsf{ufs}}([\mathbb{A}]C)$, 2 marks whether a state is final; $\mathcal{P}_{\mathsf{ufs}}([\mathbb{A}]C)$ is the set of *binding transitions* from the state x, i.e. where the input letter is stored for later use; and $\mathcal{P}_{\mathsf{ufs}}(\mathbb{A} \times C)$ is the set of transitions that compare the input letter to a previously stored one. Let $\mathbb{A}^{\#n}$ be the nominal set of n-tuples with distinct components, i.e.

$$\mathbb{A}^{\#n} = \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid |\{a_1, \ldots a_n\}| = n\}.$$

Then $\mathbb{A}^{\#n}$-pointed RNNAs accept nominal languages, i.e. equivariant maps $L\colon \mathbb{A}^* \to 2$, whose support has a cardinality of at most n [24, Corollary 5.5], under both language semantics considered in *op. cit.* Note that it is important not to restrict I to be the terminal object; in fact, that would restrict inital objects to have empty support, which may not be desirable in applications.

(5) An alternative approach to bisimulation of transition systems via so called *open maps* was introduced by Joyal, Nielsen, and Winskel [16]. There, one considers functors of type $J\colon \mathscr{P} \to \mathscr{C}$ from a small category $\mathscr{P}$ of "paths" or "linear systems" to the category $\mathscr{C}$ of "all systems" under consideration. This functor J defines a notion of *open map*. We do not recall the definition as it is irrelevant here; for details and the definition of open map see *op. cit.* The objects in $\mathscr{C}$ are usually defined as systems with an initial state, and morphisms in $\mathscr{C}$ are maps between systems that preserve (but not necessarily reflect) outgoing transitions of states, whereas the open maps in $\mathscr{C}$ are morphisms that do reflect the outgoing transitions of states that are in the reachable part of the system. Let $|\mathscr{P}|$ denote the set of objects of $\mathscr{P}$. It was shown by Lasota [18] that the canonical functor $\mathscr{C}(J(-),(-))\colon \mathscr{C} \to \mathsf{Set}^{|\mathscr{P}|}$ sends open maps in $\mathscr{C}$ to F-coalgebra homomorphisms for the following functor

$$F\colon \mathsf{Set}^{|\mathscr{P}|} \to \mathsf{Set}^{|\mathscr{P}|} \quad \text{given by} \quad (X_P)_{P \in \mathscr{P}} \mapsto \Big(\prod_{Q \in |\mathscr{P}|} \mathcal{P}(X_Q)^{\mathscr{P}(P,Q)}\Big)_{P \in \mathscr{P}}.$$

If $\mathscr{P}$ has an initial object $0_{\mathscr{P}}$ that is preserved by J, then the subcategory of $\mathscr{C}$ formed by all open maps embeds into the category of I-pointed F-coalgebras [27], where

$$I \in \mathsf{Set}^{|\mathscr{P}|} \qquad \text{with} \qquad I_P = \begin{cases} 1 & \text{if } P = 0_{\mathscr{P}} \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that once again I is not the terminal object of $\mathsf{Set}^{|\mathscr{P}|}$.

Our overall setting is that of a category $\mathscr{C}$ equipped with a *factorization system* $(\mathscr{E}, \mathscr{M})$, i.e. (1) $\mathscr{E}$ and $\mathscr{M}$ are classes of morphisms of $\mathscr{C}$ that are closed under composition with isomorphisms, (2) every morphism f of $\mathscr{C}$ has a factorization $f = m \cdot e$ with $m \in \mathscr{M}$ and $e \in \mathscr{E}$, and (3) the following *unique diagonal fill-in* property holds: for every commutative square

$$\begin{array}{ccc} A & \xrightarrow{\;e\;} & B \\ {\scriptstyle f}\big\downarrow & \overset{d}{\swarrow} & \big\downarrow{\scriptstyle g} \\ C & \xrightarrow[\;m\;]{} & D \end{array}$$

with $e \in \mathscr{E}$ and $m \in \mathscr{M}$ there exists a unique morphism $d\colon B \to C$ such that $m \cdot d = g$ and $d \cdot e = f$. We will denote morphisms in $\mathscr{M}$ by $\rightarrowtail$ and those in $\mathscr{E}$ by $\twoheadrightarrow$. While not necessary, in typical examples $\mathscr{E}$ is a class of epimorphisms, and $\mathscr{M}$ is a class of monomorphisms: (regular epi, mono) in regular categories, (epi, strong mono) in quasitoposes, (epi, mono) in toposes, etc.

We shall later only assume that $\mathscr{M}$ is a class of monomorphisms. Whenever we speak of a *subobject* of some object X we mean one that is represented by a morphism $m\colon S \rightarrowtail X$ in $\mathscr{M}$. Moreover, we shall speak of "the subobject m", i.e. we use representatives to refer to subobjects. The subobjects of an object X form a partially ordered class

$$\mathsf{Sub}(X)$$

in the usual way: for $m\colon S \rightarrowtail X$ and $m'\colon S' \rightarrowtail X$ we have $m \leq m'$ if there exists $i\colon S \to S'$ with $m' \cdot i = m$.

Remark 2.3. $(\mathscr{E}, \mathscr{M})$-factorization systems have many properties known from surjective and injective maps on Set (see [2, Chapter 14]):

(1) $\mathscr{E} \cap \mathscr{M}$ is the class of isomorphisms of $\mathscr{C}$.

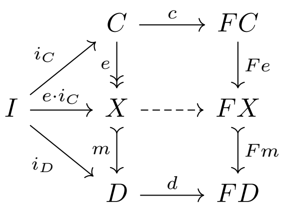
(2) $\mathscr{M}$ is stable under pullbacks.

(3) If $f \cdot g \in \mathscr{M}$ and $f \in \mathscr{M}$, then $g \in \mathscr{M}$.

(4) $\mathscr{E}$ and $\mathscr{M}$ are closed under composition.

Remark 2.4. (1) *Subcoalgebras* of pointed coalgebras are understood to be formed w.r.t. the class $\mathscr{M}$, i.e. a subcoalgebra is represented by a homomorphism $m\colon (S, s, i_S) \rightarrowtail (C, c, i_C)$ with $m \in \mathscr{M}$. Similarly, a *quotient coalgebra* is represented by a homomorphism $q\colon (C, c, i_C) \twoheadrightarrow (Q, q, i_Q)$ with $q \in \mathscr{E}$.

(2) Suppose that $F\colon \mathscr{C} \to \mathscr{C}$ preserves $\mathscr{M}$-morphisms, i.e. $Fm \in \mathscr{M}$ for every $m \in \mathscr{M}$. Then the factorization system $(\mathscr{E}, \mathscr{M})$ lifts to $\mathsf{Coalg}_I(F)$ as follows. For every homomorphism $h\colon (C, c, i_C) \to (D, d, i_D)$ one takes its factorization $h = m \cdot e$ in $\mathscr{C}$ and then obtains a unique coalgebra structure such that e and m are homomorphisms of I-pointed coalgebras using the unique diagonal fill-in property:



Definition 2.5 (Reachable coalgebra [4]). An I-pointed coalgebra (C, c, i_0) is called *reachable* if it has no proper pointed subcoalgebra, i.e. every homomorphism $m\colon (C', c', i_{C'}) \rightarrowtail (C, c, i_C)$ of I-pointed coalgebras with $m \in \mathscr{M}$ is an isomorphism.

Remark 2.6. When $I = 0$ is the initial object and $\mathscr{M}$ is a class of monomorphisms, then a coalgebra is reachable if and only if it is a quotient coalgebra of $(0, u, \mathsf{id}_0)$ where $u\colon 0 \to F0$ is the unique morphism. Indeed, suppose that (C, c, i_C) is reachable, let $h\colon (0, u, \mathsf{id}_0) \to (C, c, i_C)$ be the unique homomorphism, and take the $(\mathscr{E}, \mathscr{M})$-factorization $h = m \cdot e$. Then m represents an I-pointed subcoalgebra of (C, c, i_C) and thus is an isomorphism.

Conversely, if $e\colon (0, u, \mathsf{id}_0) \twoheadrightarrow (C, c, i_C)$ is a quotient coalgebra and $m\colon (S, s, i_S) \rightarrowtail (C, c, i_C)$ is any subcoalgebra then $e = m \cdot h$ where $h\colon 0 \to S$ is the unique morphism. By the unique diagonalization property, we obtain $d\colon C \to S$ such that $m \cdot d = \mathsf{id}_C$. Thus m is a split epimorphism and a monomorphism, whence an isomorphism. Consequently, (C, c, i_C) is reachable.

Finally, it follows that if the unique morphisms $0 \to X$ are in $\mathscr{M}$ for every object X, then $(0, u, \mathsf{id}_0)$ is the only reachable 0-pointed coalgebra.

Example 2.7. (1) For a pointed graph, reachability is clearly the usual graph theoretic concept: a pointed coalgebra (V, a, v_0) for $\mathcal{P}$ is reachable if and only if every of its nodes can be reached by a directed path from the distinguished node v_0.

(2) A deterministic automaton regarded as a pointed coalgebra for $FX = \{0, 1\} \times X^\Sigma$ on Set is reachable if and only if every of its states is reachable in finitely many steps from its initial state. This is not difficult to see directly, but it follows immediately from Theorem 4.6.

3 Functors preserving intersections

We shall see in Section 5 that the central assumption for our constructions of the reachable part of a pointed F-coalgebra is equivalent to the functor F preserving intersections. For set functors we discuss this condition in the present section. Indeed, it is an extremely mild condition satisfied by many set functors of interest:

Example 3.1. The collection of set functors which preserve intersections is closed under products, coproducts, and composition. Consequently, every polynomial endofunctor on Set preserves intersections. Moreover it is easy to see that the power set functor $\mathcal{P}$, the bag functor $\mathcal{B}$ mapping every set X to the set of finite multisets on X, as well as the functor $\mathscr{D}$ mapping X to the set of (countably supported) probability measures on X preserve intersections.

Among the finitary set functors "essentially" all functors preserve intersections. This follows from the results of Trnková on set functors as we shall now explain. First, recall that a functor is called *finitary*, if it preserves filtered colimits. For a set functor F this is equivalent to being *finitely bounded* [5, Corollary 3.11], which is the following condition: for every element $x \in FX$ there exists a finite subset $M \subseteq X$ such that $x \in Fi[FM]$, where $i\colon M \hookrightarrow X$ is the inclusion map.

Secondly, as shown by Trnková [25], every set functor preserves finite non-empty intersections. Moreover, she proved that one can turn every set functor into one that preserves all finite intersections by a simple modification at the empty set:

Proposition 3.2 (Trnková [26]). *For every set functor F there exists an essentially unique set functor $\bar{F}$ which coincides with F on non-empty sets and functions and preserves finite intersections (whence monomorphisms).*

For the proof see [26, Propositions III.5 and II.4]; for a more direct proof see Adámek and Trnková [7, III.4.5]. We call the functor $\bar{F}$ the *Trnková hull* of F.

Remark 3.3. (1) In fact, Trnková gave a construction of $\bar{F}$: she defined $\bar{F}\emptyset$ as the set of all natural transformations $C_{01} \to F$, where C_{01} is the set functor with $C_{01}\emptyset = \emptyset$ and $C_{01}X = 1$ for all non-empty sets X, and $\bar{F}e$, for the empty map $e\colon \emptyset \to X$ with $X \neq \emptyset$, maps a natural transformation $\tau\colon C_{01} \to F$ to the element given by $\tau_X\colon 1 \to FX$.

(2) There is also a different construction of $\bar{F}$ due to Barr [9]: consider the two functions $t, f\colon 1 \hookrightarrow 2$. Their intersection is the empty function $e\colon \emptyset \to 1$. Since $\bar{F}$ must preserve this intersection it follows that $\bar{F}e$ is monic and forms (not only a pullback but also) an equalizer of $\bar{F}t = Ft$ and $\bar{F}f = Ff$. Thus $\bar{F}$ must be defined on $\emptyset$ (and e) as the equalizer

$$\bar{F}\emptyset \xrightarrow{\bar{F}e} \bar{F}1 = F1 \underset{Ff}{\overset{Ft}{\rightrightarrows}} F2,$$

and on all non-empty functions f, one defines $\bar{F}f = Ff$.

(3) Trnková proved that $\bar{F}$ defines a set functor preserving finite intersections. From the proof in *op. cit.* it also follows that if F is finitary, so if $\bar{F}$.

(4) Furthermore, $\bar{F}$ is a reflection of F into the full subcategory of the category of all endofunctors on Set given by those endofunctors preserving finite intersections. That means there is a natural transformation $r\colon F \to \bar{F}$ such that for every natural transformation $s\colon F \to G$ where $G : \mathsf{Set} \to \mathsf{Set}$ preserves finite intersections there exists a unique natural transformation $s^\sharp\colon \bar{F} \to G$ such that $s^\sharp \cdot r = s$ (see [1, Corollary VII.2] for details).

(5) Finally, note that the categories of coalgebras for F and its Trnková hull $\bar{F}$ are clearly isomorphic.

For the following fact, see e.g. Adámek et al. [3, Proof of Lemma 8.8]; we include the proof for the convenience of the reader.

Corollary 3.4. *The Trnková hull of a finitary set functor preserves all intersections.*

Proof. Let F be a finitary set functor. Since $\bar{F}$ is finitary and preserves finite intersections, for every element $x \in \bar{F}X$, there exists a *least* finite set $m\colon Y \hookrightarrow X$ with x contained in $\bar{F}m$. Preservation of all intersections now follows easily: given subsets $v_i\colon V_i \hookrightarrow X$, $i \in I$, with x contained in the image of $\bar{F}v_i$ for each i, then x also lies in the image of the finite set $v_i \cap m$, hence $m \subseteq v_i$ by minimality. This proves $m \subseteq \bigcap_{i\in I} v_i$, thus, x lies in the image of $\bar{F}(\bigcap_{i\in I} v_i)$, as required. □

Remark 3.5. Note that the argument in Corollary 3.4 can be generalized to locally finitely presentable categories, see e.g. Adámek and Rosický [6] for the definition. In fact, let $\mathscr{C}$ be a locally finitely presentable category in which additionally every finitely generated object only has a finite number of subobjects; for example, Set or the categories of nominal sets (see Example 5.4(4)), of posets, and of graphs.

Then every finitary endofunctor F on $\mathscr{C}$ preserving finite intersections preserves all intersections. Indeed, since F is finitary, for every monomorphism $m\colon Y \rightarrowtail FX$ with Y finitely generated there exists a subobject $z\colon Z \rightarrowtail X$ with Z finitely generated such that m factorizes through Fz, i.e. there exists some $g\colon Y \to FZ$ such that $Fm \cdot g = f$ (see e.g. [5]). Since F preserves finite intersections it follows that there is a *least* subobject $z\colon Z \rightarrowtail X$ such that m factorizes through Fz. Indeed, let z be the intersection of all subobjects $z'\colon Z \rightarrowtail X$ such that m factorizes through Fz'. This intersection is equal to the one of all $z' \cap z$, which is a finite intersection by our hypothesis since Y is a finitely generated object. Since F preserves the latter finite intersection we obtain a morphism $g\colon Y \to FZ$ such that $Fz \cdot g = m$.

Preservation of all intersections now follows easily. Given subobjects $v_i\colon V_i \rightarrowtail X$, $i \in I$, and $m\colon Y \rightarrowtail FX$, with $m \leq Fv_i$ for all $i \in I$. We first assume that Y is finitely generated. Take the least $z\colon Z \rightarrowtail X$ such that m factorizes through Fz, i.e. $m \leq Fz$. Then we have $m \leq F(v_i \cap z) \leq Fv_i$ for all $i \in I$, where the first inclusion uses that F preserves finite intersections. Furthermore, m factorizes through Fv_i, and therefore $z \leq v_i$ by minimality for every $i \in I$. Thus, $m \leq Fz \leq F(\bigcap_{i\in I} v_i)$ as desired.

For arbitrary $m\colon Y \to FX$ write Y as the directed union of all its subobjects $s_j\colon Y_j \rightarrowtail Y$ with Y_j finitely generated. Then every s_j is contained in $F(\bigcap_{i\in I} v_i)$ by the previous argument, and therefore so is their union m.

Let us conclude this section by coming back to coalgebras to note that the condition that F preserves intersection is significant for us because it entails that every F-coalgebra has a *reachable part*, i.e. a unique reachable subcoalgebra. Indeed, recall [4] that for an intersection preserving endofunctor F on a category $\mathscr{C}$ with intersections a reachable subcoalgebra can be obtained as the intersection of all subcoalgebras of (C, c, i_C). Moreover, this intersection is the unique reachable subcoalgebra of (C, c, i_C). Given two reachable subcoalgebras S_1 and S_2 of (C, c, i_C) their intersection forms an I-pointed subcoalgebra of S_1 and S_2 and so must be isomorphic to both, thus $S_1 \cong S_2$.

4 Canonical Graphs

Note that for a given functor $F\colon \mathsf{Set} \to \mathsf{Set}$ one may define for every set X a map $\tau_X\colon FX \to \mathcal{P}X$ by

$$\tau_X(t) = \{x \in X \mid 1 \xrightarrow{t} FX \text{ does not factorize through } F(X\backslash\{x\}) \xrightarrow{Fi} FX\}, \quad (4.1)$$

where $i\colon X \setminus \{x\} \hookrightarrow X$ denotes the inclusion map.

Intuitively, $\tau_X(t)$ is the set of elements of X that occur in t.

Definition 4.1 (Gumm [13])**.** The *canonical graph* of a coalgebra $c\colon C \to FC$ is the graph given by

$$C \xrightarrow{c} FC \xrightarrow{\tau_C} \mathcal{P}C.$$

Note that for an I-pointed coalgebra (C, c, i_C) its canonical graph is I-pointed by $i_C\colon I \to I$, too.

Example 4.2. For the functor $FX = \{0,1\} \times X^\Sigma$, we have for every $i \in \{0,1\}$ and $t\colon \Sigma \to X$ that

$$\tau_X(i,t) = \{t(s) \mid s \in \Sigma\}.$$

Hence, the canonical graph of a deterministic automaton considered as an F-coalgebra is precisely its usual state transition graph (forgetting the labels of transitions and the finality of states).

Lemma 4.3 (Gumm [13, Theorem 7.4])**.** *If $F\colon \mathsf{Set} \to \mathsf{Set}$ preserves intersections, then the above maps $\tau_X\colon FX \to \mathcal{P}X$ form a* sub-cartesian transformation*, i.e. for every injective map $m\colon X \rightarrowtail Y$ the following diagram is a pullback square:*

$$\begin{array}{ccc} FX & \xrightarrow{\tau_X} & \mathcal{P}X \\ {\scriptstyle Fm}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathcal{P}m} \\ FX & \xrightarrow{\tau_Y} & \mathcal{P}X \end{array} \qquad (4.2)$$

Conversely, if τ is a sub-cartesian transformation, then F preserves intersections.[1]

[1]For this converse, Gumm assumed that F preserves monomorphisms; however, this is not needed since $\mathcal{P}$ preserves monomorphisms and monomorphisms are stable under pullback.

Theorem 4.4 (Gumm [13, Theorem 8.1]). *Assume that F preserves inverse images and intersections. Then $\tau\colon F \to \mathcal{P}$ is a natural transformation.*

Example 4.5. To see that τ is not a natural transformation in general, one may consider the functor $R\colon \mathsf{Set} \to \mathsf{Set}$ defined by $RX = \{(x, y) \in X \times X : x \neq y\} + \{*\}$ on sets X and for a function $f\colon X \to Y$ put

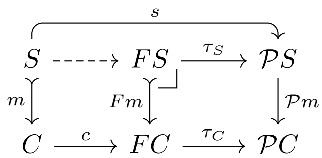
$$Rf(*) = * \quad \text{and} \quad Rf(x, y) = \begin{cases} * & \text{if } f(x) = f(y) \\ (f(x), f(y)) & \text{else.} \end{cases}$$

Now let $X = \{0, 1\}$, $Y = \{0\}$, and $f\colon X \to Y$ the evident function. Then $(0, 1) \in RX$, and $\tau_X(0, 1) = X$. Furthermore, $\mathcal{P}f(X) = Y$. But $Rf(0, 1) = *$, and $\tau_Y(*) = \emptyset$.

Our observation in this section is that reachability of a coalgebra and its canonical graph are equivalent concepts:

Theorem 4.6. *Let $F\colon \mathsf{Set} \to \mathsf{Set}$ preserve intersections. Then an I-pointed coalgebra for F is reachable if and only if so is its canonical graph.*

Proof. Let (C, c, i_C) be an I-pointed F-coalgebra. Then we see that subcoalgebras of (C, c, i_C) are in one-to-one correspondence with subgraphs of the canonical graph. Indeed, given any subcoalgebra $m\colon (S, s, i_S) \rightarrowtail (C, c, i_C)$, we have that $(S, \tau_S \cdot s, i_S)$ is an I-pointed subgraph of $(C, \tau_C \cdot c, i_C)$ via m due to the commutativity of (4.2). Conversely, let (S, s, i_S) be an I-pointed subgraph of the canonical graph $(C, \tau_C \cdot c, i_C)$ via the monomorphism $m\colon S \rightarrowtail C$, say. Then, using that (4.2) is a pullback, we obtain an F-coalgebra structure on S turning it into a subcoalgebra of (C, c, i_C):



We conclude that (C, c, i_C) does not have any proper subcoalgebra w.r.t. F if and only if its canonical pointed graph $(C, \tau_C \cdot c, i_C)$ does not have a proper subcoalgebra w.r.t $\mathcal{P}$. As we saw in Example 2.7(1), the latter is equivalent to that I-pointed graph being reachable, which completes the proof. $\square$

5 Iterative Construction

This section is devoted to a new iterative construction of the reachable part of a given I-pointed coalgebra (C, c, i_C), the unique reachable subcoalgebra of (C, c, i_C), reminiscent of breadth-first search for graphs.

Assumption 5.1. Throughout this section we assume that the base category $\mathscr{C}$ has arbitrary (small) coproducts, is well-powered and is equipped with an $(\mathscr{E}, \mathscr{M})$-factorization system, where $\mathscr{M}$ is a class of monomorphisms.

Remark 5.2. We collect a number of easy consequence of Assumption 5.1.

(1) Note that the above assumptions imply that $\mathscr{C}$ has all unions, i.e. for every object C of $\mathscr{C}$ the partially ordered set $\mathsf{Sub}(C)$ of its subobjects has all joins. In fact, given a family $(m_i \colon C_i \rightarrowtail C)_{i\in I}$, their union m is given by the following $(\mathscr{E}, \mathscr{M})$-factorization:

$$\coprod_{i\in I} C_i \xrightarrow{\;e\;} \bigcup_{i\in I} m_i \xrightarrow{\;m\;} C, \qquad [m_i]_{i\in I} = m \cdot e \colon \coprod_{i\in I} C_i \to C.$$

(2) It follows that $\mathsf{Sub}(C)$ is a complete lattice, and therefore that $\mathscr{C}$ has all intersections. Moreover, we show that intersections are given by pullbacks (even though we did not assume their existence). In fact, given the family $(m_i \colon C_i \rightarrowtail C)_{i\in I}$ take their intersection, i.e. their meet, $m \colon M \rightarrowtail C$ in $\mathsf{Sub}(C)$. The morphisms $p_i \colon M \rightarrowtail C_i$ witnessing $m \leq m_i$ yield the projections of the (wide) pullback. Moreover, given any compatible cone $f_i \colon X \to C_i$ such that $m_i \cdot f_i = m_j \cdot f_j$ for all $i, j \in I$ take the $(\mathscr{E}, \mathscr{M})$-factorization $n \cdot e$ of that morphism and use the diagonal fill-in property

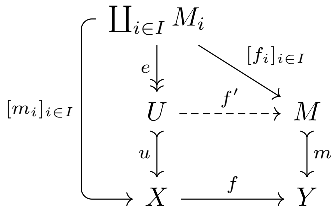
$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & X' \\ {\scriptstyle f_i}\downarrow & \swarrow & \downarrow{\scriptstyle n} \\ C_i & \xrightarrow[\;m_i\;]{} & C \end{array}$$

in order to see that $n \leq m_i$ for all $i \in I$. Thus, we have $n \leq m$, which is witnessed by a (necessarily unique) morphism $h \colon X' \rightarrowtail M$ such that $n \cdot h = m$. Then $h \cdot e$ is the desired unique factorizing morphism showing that M is a wide pullback of the m_i.

(3) In addition, using the well-poweredness of $\mathscr{C}$ we see that it has *preimages*, i.e. pullbacks along morphisms in $\mathscr{M}$. Indeed, suppose we are given a morphism $f \colon X \to Y$ and a subobject $m \colon M \rightarrowtail Y$. Then we form the family of all subobjects $m_i \colon M_i \rightarrowtail X$ for which there exists a restricting morphism $f_i \colon M_i \to M$, i.e. $f \cdot m_i = m \cdot f_i$, and we take their union:

$$(u \colon U \rightarrowtail X) := \bigcup \{ m_i \colon M_i \rightarrowtail X \mid \exists f_i \colon M_i \to M \text{ with } f \cdot m_i = m \cdot f_i \} \quad (5.1)$$

Using the diagonal fill-in property, we obtain a morphism $f' \colon U \to M$ such that the following diagram commutes:



In order to show that the lower square is a pullback, suppose that we have morphisms $p\colon Z \to X$ and $q\colon Z \to M$ with $f \cdot p = m \cdot q$. Take the $(\mathscr{E},\mathscr{M})$-factorization $p = (Z \overset{e'}{\twoheadrightarrow} I \overset{m'}{\rightarrowtail} X)$. Then $(f \cdot m') \cdot e' = m \cdot q$. Hence, by the unique diagonal fill-in property, we obtain some $d\colon I \to M$ such that $m \cdot d = f \cdot m'$ and $d \cdot e' = q$. Thus, $m'\colon I \rightarrowtail X$ is one of the subobjects m_i in (5.1), and therefore $m' \le \bigcup m_i = u$, i.e. we have a morphism $s\colon I \to P$ with $u \cdot s = m'$. Then $h := s \cdot e'\colon Z \to U$ is the desired factorization of p, q. Indeed, we have

$$u \cdot (s \cdot e') = m' \cdot e' = p,$$

and to see that $f' \cdot h = q$ we use that m is a monomorphism and compute

$$m \cdot f' \cdot h = f \cdot u \cdot h = f \cdot p = m \cdot q.$$

Remark 5.3. In the following example and in Section 6 we shall mention Kleisli categories. Recall that the Kleisli category $\mathcal{K}\ell(T)$ for a monad (T, μ, η) on $\mathscr{C}$ has the same objects as $\mathscr{C}$ and hom-sets $\mathcal{K}\ell(T)(X,Y) = \mathscr{C}(X, TX)$. We use the notation $f\colon X \multimap Y$ to denote a morphism $f \in \mathcal{K}\ell(T)(X,Y)$, and we call such morphisms *Kleisli morphisms*. The composition of Kleisli morphisms $f\colon X \multimap Y$ and $g\colon Y \multimap Z$ is denoted by $g \circ f$ and defined by

$$g \circ f = (X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{\mu_Z} TZ).^2$$

The identity morphism on X is given by the unit $\eta_X\colon X \to TX$ of the monad.

Example 5.4. (1) Recall that every complete category $\mathscr{C}$ is equipped with a (strong epi, mono)-factorization system and with an (epi, strong mono)-factorization system [2, Theorems 14.17 and 14.19].

Hence, every complete and well-powered category $\mathscr{C}$ with coproducts meets Assumption 5.1.

(2) The category Rel of sets and relations has all coproducts and a factorization system given by

$$\mathscr{E} = \text{all surjective relations}, \qquad \text{and} \qquad \mathscr{M} = \text{all injective maps}.$$

Note that Rel is the Kleisli category of the power-set monad.

[2]Note that in terms of the Kleisli extension $g^* = \mu_Z \cdot Tg$ we have that $g \circ f = g^* \cdot f$.

(3) A similar factorization system can be obtained for stochastic relations which give for a point in a set X a probability distribution over the points in another set Y (in lieu of a set of points in Y). These stochastic relations are given by morphisms $X \to \mathcal{D}Y$ in the Kleisli category of the distribution monad $\mathcal{D}$ on Set. This monad is given as a Kleisly triple $(\mathcal{D}, \eta, (-)^*)$ as follows: for every set X we have

$$\mathcal{D}X = \{f\colon X \to [0,1] \mid \sum_{x \in X} f(x) = 1\}$$

(note that the above sum necessarily has at most countably many non-zero summands) and $\eta_X\colon X \to \mathcal{D}X$ given by the Dirac distribution

$$\eta_X(x)(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{else.} \end{cases}$$

The Kleisli extension of a map $h\colon X \to \mathcal{D}Y$ is the map $h^*\colon \mathcal{D}X \to \mathcal{D}Y$ given by

$$h^*(f)(y) = \sum_{x \in X} f(x) \cdot h(x)(y).$$

The Kleisli category $\mathcal{K}\ell(\mathcal{D})$ has all coproducts and a factorization system given by the following two classes of morphisms:

$$\begin{aligned} \mathscr{E} &= \{e\colon X \to \mathcal{D}Y \mid \forall y \in Y\, \exists x \in X\colon e(x)(y) \neq 0\}, \text{and} \\ \mathscr{M} &= \{m\colon X \to \mathcal{D}Y \mid m = \eta_Y \cdot m' \text{ for some injective map } m'\colon X \to Y\}. \end{aligned}$$

Hence, the class $\mathscr{M}$ consists essentially of injective maps considered as morphisms in $\mathcal{K}\ell(\mathcal{D})$. It is easy to see that the two classes of morphisms are closed under composition, and that every morphism in $\mathcal{K}\ell(\mathcal{D})$ has an essentially unique $(\mathscr{E}, \mathscr{M})$-factorization, given by the least bounds of $\mathscr{D}$. Moreover, $\mathscr{E}$ and $\mathscr{M}$ contain all isomorphisms of $\mathcal{K}\ell(\mathcal{D})$. In fact, we show below that those isomorphisms correspond precisely to bijective maps; more precisely, a morphism $h\colon X \to \mathcal{D}Y$ is an isomorphism if and only if $h = \eta_Y \cdot h'$ for some bijective map $h'\colon X \to Y$. It follows that $(\mathscr{E}, \mathscr{M})$ is a factorization system [2, Theorem 14.7].

We proceed to prove the above characterization of isomorphisms in $\mathcal{K}\ell(\mathcal{D})$, i.e. we show that $h\colon X \to \mathcal{D}(Y)$ is an isomorphism if and only if for every $x \in X$ there exists a $y \in Y$ with $h(x)(y) = 1$ and for every $y \in Y$ there is exists a unique $x \in X$ with $h(x)(y) > 0$ (and hence $h(x)(y) = 1$).

The 'if' direction holds, because the first condition states that $h = \eta_Y \cdot h'$ for some map $h'\colon X \to Y$ and the second condition states that h' is bijective.

For the 'only if' direction suppose that $g\colon Y \to \mathcal{D}X$ is inverse to h. Observe that for every $x \in X$ there is some $y \in Y$ with $g(y)(x) > 0$, because $1 = (g \circ h)(x)(x) = \sum_{y \in Y} g(y)(x) \cdot h(x)(y)$, and similarly for every $y \in Y$ there is some $x \in X$ with $h(x)(y) > 0$. By the definition of $\mathcal{D}X$, there is some $x \in X$ with $g(y)(x) \neq 0$, and similarly, for every $x \in X$ there is some $y \in Y$ with $h(x)(y) > 0$.

To verify the first condition, let $x \in X$ and $y_1, y_2 \in Y$ with $h(x)(y_1) \neq 0 \neq h(x)(y_2)$. By the previous observation, there is some $y \in Y$ with $g(y)(x) > 0$, and therefore $(h \circ g)(y)(y_1) \neq 0 \neq (h \circ g)(y)(y_2)$. Hence, $y_1 = y = y_2$ since $h \circ g$ is the identity morphism η_Y in $\mathcal{K}\ell(\mathcal{D})$. So for every $x \in X$ there is a unique $y \in Y$ with $h(x)(y) > 0$ and therefore $h(x)(y) = 1$.

For the second condition, we have already observed that for every $y \in Y$ there exists some $x \in X$ with $h(x)(y) > 0$. For the uniqueness, let $y \in Y$, $x_1, x_2 \in X$ with $h(x_1)(y) \neq 0 \neq h(x_2)(y)$. Let $x \in X$ be such that $g(y)(x) > 0$. Then $(g \circ h)(x_1)(x) \neq 0 \neq (g \circ h)(x_2)(x)$, and hence $x_1 = x = x_2$ since $g \circ h$ is the identity morphism η_X in $\mathcal{K}\ell(\mathcal{D})$.

(4) Let $\mathbb{S} = (S, +, \cdot, 0, 1)$ be a semiring. Then similarly as in point (3) we obtain a monad $\mathbb{S}^{(-)}$ on Set given as a Kleisli triple as follows: for every set X, we have

$$\mathbb{S}^{(X)} = \{f \colon X \to S \mid f(x) \neq 0 \text{ for finitely many } x \in X\},$$

and the unit $\eta_X \colon X \to \mathbb{S}^{(X)}$ and the Kleisli lifting are defined precisely as in point (3).

We also consider the same classes $\mathscr{E}$ and $\mathscr{M}$ as in the previous point (3), and they can be shown to form a factorization system provided that the given semiring fulfils the following conditions: $(S, +, 0)$ and $(S, \cdot, 1)$ are positive monoids, i.e. whenever $a + b = 0$ then $a = 0$ or $b = 0$, and similarly for the multiplication and 1, and the semiring is zero-divisor-free, i.e. whenever $a \cdot b = 0$ then $a = 0$ or $b = 0$.

Our construction of the reachable part is based on the following notion capturing the part of an object Y that is actually used by a morphism $f \colon X \to FY$. For the class of all monomorphisms this notion was introduced by Alwin Block [10] under the name "base":

Definition 5.5. Given a functor $F \colon \mathscr{C} \to \mathscr{D}$ and a class $\mathscr{M}$ of monomorphisms of $\mathscr{C}$. We say that F *has least bounds (w.r.t. $\mathscr{M}$)* if for every morphism $f \colon X \to FY$ there is a *least* morphism $m \colon Z \rightarrowtail Y$ in $\mathscr{M}$ such that f factorizes through Fm. This means, there exists some $g \colon X \to FZ$ with

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & FY \\ & {\scriptstyle g}\searrow & \uparrow{\scriptstyle Fm} \\ & & FZ \end{array} \qquad \text{in } \mathscr{D},$$

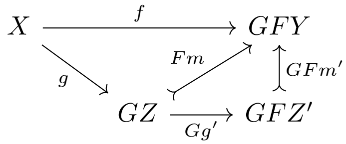
and for every $m' \colon Z' \to Y$ in $\mathscr{M}$ and $g' \colon X \to FZ'$ with $Fm' \cdot g' = f$ there exists a (necessarily unique) $h \colon Z \to Z'$ with $m' \cdot h = m$, i.e. $m \leq m'$ in $\mathsf{Sub}(Y)$.

The triple (Z, g, m) is called the *bound of f*, and the above triple (Z', g', m') is said to *compete* with the bound.

Proposition 5.6. *Functors having least bounds are closed under composition.*

Proof. Let $F \colon \mathscr{C} \to \mathscr{D}$ and $G \colon \mathscr{D} \to \mathscr{D}'$ have least bounds w.r.t the classes $\mathscr{M}$ and $\mathscr{M}'$ of $\mathscr{C}$ and $\mathscr{D}$, respectively. We will prove that GF has least bounds w.r.t. $\mathscr{M}$.

In order to see this consider a morphism $f\colon X \to GFY$. Then first take its bound w.r.t. G to obtain $g\colon X \to GFZ$ and $m\colon Z \rightarrowtail FY$ in $\mathscr{M}'$ such that $Gm \cdot g = f$, and then take the bound of m w.r.t. F to obtain $g'\colon Z \to FZ'$ and $m'\colon Z' \rightarrowtail Y$ in $\mathscr{M}$ such that $Fm' \cdot g' = g$:



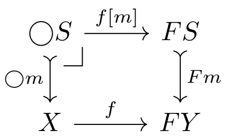
Then $Gg' \cdot g$ and m' form the desired bound of f w.r.t. GF. Indeed, given any $g''\colon X \to GFZ''$ and $m''\colon Z'' \rightarrowtail GFY$ with $GFm'' \cdot g'' = f$ one first uses minimality of the bound (Z, g, m) to obtain some $h\colon Z \to FZ''$ with $Gm'' \cdot h = m$, and then one uses the minimality of (Z', g', m') w.r.t. F to obtain $h'\colon Z' \to Z''$ such that $m'' \cdot h' = m'$ as required. $\square$

Gumm [13, Corollary 4.8] proved that, in the case where F is an endofunctor on Set and $\mathscr{M}$ is the class of all monomorphisms, F has least bounds if and only if it preserves intersections. We now provide the proof in our setting, and we slightly extend the result by a statement involving the following operator, which extends the "next time" operator of Jacobs [15] for coalgebras to arbitrary morphisms:

Definition 5.7. Suppose that $F\colon \mathscr{C} \to \mathscr{C}$ preserves $\mathscr{M}$-morphism, i.e. Fm lies in $\mathscr{M}$ for every m in $\mathscr{M}$. For every morphism $f\colon X \to FY$ we define the operator

$$\bigcirc_f\colon \mathsf{Sub}(Y) \to \mathsf{Sub}(X)$$

as follows (we drop the subscript f whenever this morphism is clear from the context): given a subobject $m\colon S \to Y$ we form the preimage of Fm under f, i.e. we form the pullback below:



Remark 5.8. Note that with our convention to take subobjects w.r.t. the class $\mathscr{M}$, a functor $F\colon \mathscr{C} \to \mathscr{C}$ *preserves intersections* if and only if it preserves $\mathscr{M}$-morphisms and (wide) pullbacks of families of $\mathscr{M}$-morphisms.

Proposition 5.9. *Let $F\colon \mathscr{C} \to \mathscr{C}$ preserve $\mathscr{M}$-morphisms. Then the following are equivalent:*

(1) *F preserves intersections.*

(2) *F has least bounds w.r.t. $\mathscr{M}$.*

(3) *For every $f\colon X \to FY$, the operator $\bigcirc_f$ has a left-adjoint.*

Note that since F preserves $\mathscr{M}$-morphisms, the factor g in Definition 5.5 is uniquely determined.

Proof. For (3) $\Longrightarrow$ (1) choose $f = \mathsf{id}_{FY}$ then $\bigcirc\colon m \mapsto Fm$ is a right-adjoint and so preserves all meets, i.e. F preserves intersections.

The converse (1) $\Longrightarrow$ (3) follows from the easily established fact that intersections are stable under preimage, i.e. for every morphism $f\colon X \to Y$ and every family $m_i\colon S_i \rightarrowtail Y$ of subobjects the intersection $m\colon P \rightarrowtail X$ of the preimages of the m_i under f yields a pullback

$$\begin{array}{ccc} P & \longrightarrow & \bigcap S_i \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle \bigcap m_i} \\ X & \xrightarrow{\;f\;} & Y \end{array}$$

Thus, if F preserves intersections, so does $\bigcirc$, whence it is a right-adjoint.

For (1) $\Longrightarrow$ (2), consider $f\colon X \to FY$ and define $m\colon Y' \rightarrowtail Y$ to be the intersection of all subobjects with the desired factorization property:

$$(m\colon Y' \rightarrowtail Y) := \bigcap \big\{ m_i\colon Y_i \rightarrowtail Y \mid \exists f_i\colon X \to FY_i \text{ with } Fm_i \cdot f_i = f \big\} \tag{5.2}$$

This intersection exists since $\mathscr{C}$ is well-powered, and it is preserved by F. The witnessing morphisms $f_i\colon X \to FY_i$ from (5.2) form a cone for this intersection, inducing a unique $f'\colon X \to FY'$ such that $Fs_i \cdot f' = f_i$ for all i, where $s_i\colon Y' \rightarrowtail Y_i$ are the morphisms witnessing $m \le m_i$, i.e. we have $m_i \cdot s_i = m$. It follows that we have

$$Fm \cdot f' = Fm_i \cdot Fs_i \cdot f' = Fm_i \cdot f_i = f,$$

whence (Y', f', m) is the desired bound of f. In fact, mimimality clearly holds: whenever $(\bar{Y}, g, \bar{m})$ competes with that triple, we see that $\bar{m}$ is contained in the set in (5.2), thus $m \le \bar{m}$.

For (2) $\Longrightarrow$ (1), consider an intersection

$$(w\colon W \rightarrowtail Z) = \bigcap \big\{ y_i\colon Y_i \rightarrowtail Z \mid i \in I \big\},$$

and let $w_i\colon W \to Y_i$, $i \in I$, denote the corresponding pullback projections. Suppose we have a competing cone

$$(c_i\colon C \to FY_i)_{i \in I} \text{ with } Fy_i \cdot c_i = Fy_j \cdot c_j \text{ for all } i, j \in I.$$

We can assume wlog that $I \neq \emptyset$, because for $I = \emptyset$, the intersection $w = \mathsf{id}_Z$ is preserved by every functor. We need to prove that there exists a unique morphism $u\colon C \to FW$ such that $c_i \cdot u = w_i$ for all $i \in I$. Using our hypothesis (2), we take the bound (Z', f', z) of $f := Fy_i \cdot c_i$ for some $i \in I$. Hence, the following diagrams commute for all $i \in I$:

$$
\begin{array}{ccc}
 & & FY_i \\
 & \overset{c_i}{\nearrow} & \Big\downarrow{\scriptstyle Fy_i} \\
C & \xrightarrow{\;f\;} & FZ \\
 & \underset{f'}{\searrow} & \Big\uparrow{\scriptstyle Fz} \\
 & & FZ'
\end{array}
$$

For every $i \in I$, the triple (Y_i, c_i, y_i) competes with the bound of f. Hence, we have, for every $i \in I$, a unique $z_i\colon Z' \rightarrowtail Y_i$ with $y_i \cdot z_i = z$. Thus, $(z_i\colon Z' \rightarrowtail Y_i)_{i \in I}$ is a competing cone for the intersection of all y_i, and so we obtain a unique morphism $v\colon Z' \to W$ such that the following triangles commute:

$$
\begin{array}{ccc}
 & & Y_i \\
 & \overset{z_i}{\nearrow} & \Big\uparrow{\scriptstyle w_i} \\
Z' & \xrightarrow{\;v\;} & W
\end{array}
\qquad \text{for every } i \in I.
$$

Furthermore, since every Fy_i lies in $\mathscr{M}$ and is therefore monomorphic, the following diagrams commute:

$$
\begin{array}{ccc}
FY_i & \overset{Fy_i}{\rightarrowtail} & FZ \\
{\scriptstyle c_i}\Big\uparrow & \overset{Fz_i}{\nwarrow} & \Big\uparrow{\scriptstyle Fz} \\
C & \xrightarrow{\;f'\;} & FZ'
\end{array}
\qquad \text{for every } i \in I.
$$

Now let $u := Fv \cdot f'\colon C \to FW$. Then the following diagram commutes:

$$
\begin{array}{ccccc}
 & & & & FY_i \\
 & \overset{c_i}{\nearrow} & & \overset{Fz_i}{\nearrow} & \Big\uparrow{\scriptstyle Fw_i} \\
C & \xrightarrow{\;f'\;} & FZ' & \xrightarrow{\;Fv\;} & FW \\
 & & \underbrace{\qquad\qquad}_{u} & &
\end{array}
\qquad \text{for every } i \in I,
$$

as desired. Finally, since Fw_i is monomorphic for every $i \in I$ and $I \neq \emptyset$, u is the unique morphism such that $Fw_i \cdot u = c_i$ for every $i \in I$. □

Assumption 5.10. In addition to Assumption 5.1 we now assume for the remainder of this section that $F\colon \mathscr{C} \to \mathscr{C}$ is a functor preserving intersections (equivalently, F has least bounds).

Remark 5.11. Note that if F is a finitary functor on $\mathscr{C} = \mathsf{Set}$ with the usual factorization system given by surjective and injective maps, then we do not need to assume that F preserves intersections. In fact, using Corollary 3.4, we may work with the Trnková hull $\bar{F}$ recalling that the category of $\bar{F}$-coalgebras is isomorphic to the category of F-coalgebras.

Example 5.12. Let us continue Example 5.4.

(1) Every intersection-preserving functor on a complete and well-powered category has least bounds. Note that this does not need the existence of coproducts; in fact, the proof of Proposition 5.9 just needs the existence of intersections in $\mathscr{C}$.

(2) It is easy to see that every functor $\bar{F}$ on Rel extending an intersection-preserving set functor F satisfies our assumptions, i.e. $\bar{F}$ preserves maps and injective ones. Moreover, given a morphism $f\colon X \to FY$ in Rel, we write it as a map $f\colon X \to \mathcal{P}FY$. Using that $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ preserves intersection and thus has least bounds, the argument of Proposition 5.6 instantiated to show that $\mathcal{P}F$ has least bounds also shows how to obtain the bound of f w.r.t. $\bar{F}$ on Rel.

(3) A similar argument holds for extended functors $\bar{F}$ on $\mathcal{K}\ell(\mathcal{D})$ and $\mathcal{K}\ell(\mathbb{S}^{(-)})$ for a semiring $\mathbb{S}$ satisfying the conditions mentioned in Example 5.4(4).

However, we will see in Section 6 that one can construct the reachable part of an $\bar{F}$-coalgebra even if one does not have a factorization system on the Kleisli category, i.e. without any further assumption on the semiring $\mathbb{S}$.

From the fact that F has least bounds we obtain for every morphism $f\colon X \to FY$ the following operator mapping subobjects of X to those of Y:

Definition 5.13. Let $f\colon X \to FY$ be a morphism. The operator

$$\ominus_f\colon \mathsf{Sub}(X) \to \mathsf{Sub}(Y)$$

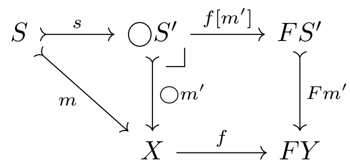
takes a subobject $m\colon S \rightarrowtail X$ to the bound of $f \cdot m$. In particular, we have the commutative square below (we omit the subscipt whenever f is clear from the context):

$$\begin{array}{ccc} S & \xrightarrow{\;g\;} & F(\ominus S) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle F(\ominus m)} \\ X & \xrightarrow{\;f\;} & FY \end{array}$$

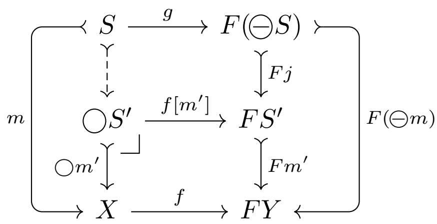
Proposition 5.14. *For every morphism $f\colon X \to FY$, the operator $\ominus$ is the left-adjoint of the "next time" operator $\bigcirc$ from Definition 5.7.*

Consequently, $\ominus$ preserves all unions, and in particular it is monotone.

Proof. Let $f\colon X \to FY$ be any morphism and assume that $m \le \bigcirc m'$ for some subobjects $m\colon S \rightarrowtail X$ and $m'\colon S' \rightarrowtail Y$. Then we have a commutative diagram



and therefore $(S', f[m'] \cdot s, m')$ is competing with the bound of $f \cdot m$. Thus, $\ominus m \le m'$. Conversely, suppose that $\ominus m \le m'$, witnessed by $j\colon \ominus S \rightarrowtail S'$. Then consider the following diagram, where $g\colon S \to F(\ominus S)$ comes from the bound of $f \cdot m$:



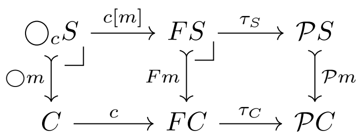
Since its outside and its right-hand part commute, we obtain the dashed arrow using the universal property of the pullback forming $\bigcirc m'$. This proves that $m \leq \bigcirc m'$ as desired. $\square$

Remark 5.15. For a coalgebra $c\colon C \to FC$, the operators $\bigcirc$ and $\ominus$ on $\mathsf{Sub}(C)$ are a generalized semantic counterpart of the next time and previous time operators, respectively, of classical linear temporal logic, see e.g. Manna and Pnüeli [19]. In fact, consider the functor $FX = \mathcal{P}(A \times X)$ on Set whose coalgebras are labelled transition systems. For a transition system $c\colon C \to \mathcal{P}(A \times C)$ and a subset $m\colon S \hookrightarrow C$ we have

$$\bigcirc S = \big\{x \in C \mid \text{for all } (a,s) \in c(x),\ s \in S\big\},$$
$$\ominus S = \big\{x \in C \mid (a,x) \in c(s) \text{ for some } a \in A \text{ and } s \in S\big\}.$$

Proposition 5.16. *For an intersection preserving set functor, $\ominus$ may be computed on the canonical graph of a given coalgebra.*

Proof. Suppose $F\colon \mathsf{Set} \to \mathsf{Set}$, which preserves intersections by Assumption 5.10. By Lemma 4.3 we have the sub-cartesian transformation $\tau_X\colon FX \to \mathcal{P}X$ as defined in (4.1). Given a coalgebra $c\colon C \to FC$, we need to prove that $\ominus_c = \ominus_{\tau_C \cdot c}$. We know that $\ominus_c$ is left-adjoint to $\bigcirc_c$ by Proposition 5.14. Similarly, $\ominus_{\tau_C \cdot c}$ is left-adjoint to $\bigcirc_{\tau_C \cdot c}$, since $\mathcal{P}$ preserves intersections. Hence, it suffices to show that $\bigcirc_c = \bigcirc_{\tau_C \cdot c}\colon \mathsf{Sub}(C) \to \mathsf{Sub}(C)$. Indeed, for every $m\colon S \rightarrowtail C$ we have that the following composition of two pullbacks



which is again a pullback diagram. Thus $\bigcirc m\colon \bigcirc_c S \rightarrowtail C$ is the pullback of $\mathcal{P}m$ along $\tau_C \cdot c$, i.e. $\bigcirc_c = \bigcirc_{\tau_C \cdot c}$. $\square$

Remark 5.17. In connection with reachable coalgebras, the operator $\ominus$ was recently used in the work of Barlocco, Kupke, and Rot [8]. Their results were obtained independently from ours but almost at the same time. They work with a complete and well-powered category $\mathscr{C}$, so $\mathscr{M}$ is the class of all monomorphisms (cf. Example 5.4(1)). First they show that every intersection preserving endofunctor F on $\mathscr{C}$ has least bounds (i.e. the implication (1) $\implies$ (2) in Proposition 5.9).

Furthermore, it is easy to see that, for every F-coalgebra (C, c), $\ominus$ is a monotone operator on $\mathsf{Sub}(C)$. In addition, we see that $\ominus$ preserves all unions. Indeed, in the setting of a complete and well-powered category, every $\mathsf{Sub}(C)$ is a complete lattice having all intersections. Thus (the proof of) Proposition 5.9 shows that $\bigcirc$ has the left-adjoint $\ominus$.

Moreover, it is shown in op. cit. that for every point $i_0 \colon 1 \to C$ the reachable part of (C, c, i_0) is the least fixed point of $i_0 \vee \ominus(-)$.

However, note that the assumption of completeness may be limiting applications, e.g. the category Rel in Example 5.4(2) is not complete.

Barlocco et al. [8] also prove the following fact. The proof is the same in our setting, and we present it here for the convenience of the reader.

Proposition 5.18. *Let $c\colon C \to FC$ be a coalgebra and $m\colon S \rightarrowtail C$ a subobject. Then S carries a subcoalgebra of (C, c) if and only if m is a prefixed point of $\ominus$.*

Proof. Suppose first that we have $\ominus m \le m$, i.e. we have some $i\colon \ominus S \rightarrowtail S$ such that $m \cdot i = \ominus m$. Then the following diagram commutes:

$$\begin{array}{ccccc} S & \xrightarrow{\;g\;} & F(\ominus S) & \xrightarrow{\;Fi\;} & FS \\ {\scriptstyle m}\downarrow & & \searrow{\scriptstyle F(\ominus m)} & & \downarrow{\scriptstyle Fm} \\ C & & \xrightarrow{\qquad c \qquad} & & FC \end{array}$$

This shows that $(S, Fi \cdot g)$ is a subcoalgebra of (C, c).

Conversely, suppose that $m\colon (S, s) \rightarrowtail (C, c)$ is a subcoalgebra. Then (S, s, m) competes with the bound $(\ominus S, g, \ominus m)$ of $c \cdot m$ and therefore we have $\ominus m \le m$. $\square$

We now present our new construction of the reachable part of an I-pointed coalgebra as the union of all iterated applications of $\ominus$ on the given I-pointing.

Construction 5.19. Given an I-pointed F-coalgebra $I \xrightarrow{i_C} C \xrightarrow{c} FC$, define subobjects $m_k\colon C_k \rightarrowtail C$, $k \in \mathbb{N}$, inductively:

(1) Let C_0 be the $(\mathscr{E}, \mathscr{M})$-factorization of i_C:

$$i_C = \left(I \xrightarrow{\;i'_C\;} C_0 \xrightarrow{\;m_0\;} C \right). \tag{5.3}$$

(2) Given $m_k\colon C_k \to C$, let $m_{k+1} = \ominus m_k \colon C_{k+1} = \ominus C_k \rightarrowtail C$, i.e. m_{k+1} is given by the bound of $c \cdot m_k$:

$$\begin{array}{ccc} C_k & \xrightarrow{\;c_k\;} & FC_{k+1} \\ {\scriptstyle m_k}\downarrow & & \downarrow{\scriptstyle Fm_{k+1}} \\ C & \xrightarrow{\;c\;} & FC \end{array} \tag{5.4}$$

We define the subobject $m\colon R \rightarrowtail C$ to be the union of all m_k, $k \in \mathbb{N}$:

$$m := \bigcup_{k\in\mathbb{N}} \ominus^k(m_0).$$

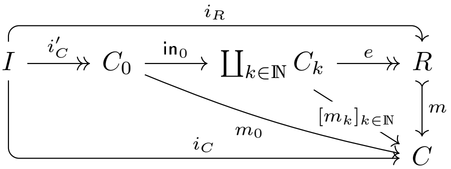
Theorem 5.20. *For every I-pointed coalgebra (C, c, i_C), the union $m\colon R \rightarrowtail C$ is a reachable subcoalgebra of (C, c, i_C).*

Proof. We will prove that R carries the structure of an I-pointed coalgebra, which is reachable, and that m is a homomorphism of I-pointed coalgebras.

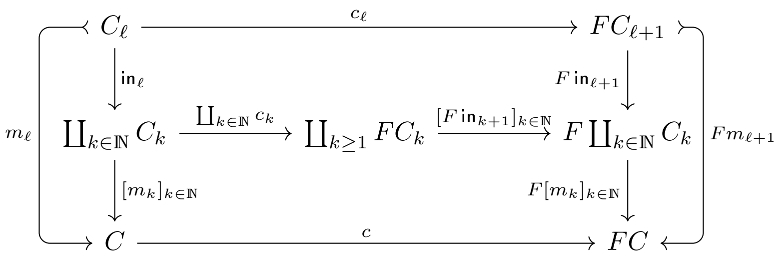
Recall from Remark 5.2(1) that the union m is formed by taking the $(\mathscr{E}, \mathscr{M})$-factorization $m \cdot e = [m_k]_{k\in\mathbb{N}}\colon \coprod_{k\in\mathbb{N}} C_k \to C$, and let us denote the coproduct injections by $\mathsf{in}_k\colon C_k \to \coprod_{k\in\mathbb{N}} C_k$.

By Remark 2.3(3), $e\cdot\mathsf{in}_k\colon C_k \rightarrowtail R$ is in $\mathscr{M}$ for all $k \in \mathbb{N}$, because $m_k = m\cdot e\cdot\mathsf{in}_k$ and m is in $\mathscr{M}$.

(1) The pointing $i_R := e \cdot \mathsf{in}_0 \cdot i'_C\colon I \to R$ is preserved by m because the following diagram commutes



For the coalgebra structure on R, first note that the outside of the following diagram commutes for every $\ell \in \mathbb{N}$ using (5.4):

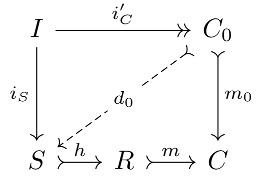


Since the coproduct injections in_ℓ form a jointly epimorphic family, it follows that the lower part commutes, i.e. we have obtained a coalgebra structure on $\coprod_{k\in\mathbb{N}} C_k$ such that $[m_k]_{k\in\mathbb{N}}$ is a coalgebra homomomorphism. Since $m\colon R \rightarrowtail C$ is obtained by the $(\mathscr{E}, \mathscr{M})$-factorization $m \cdot e = [m_k]_{k\in\mathbb{N}}$, we also obtain a coalgebra structure $r\colon R \to FR$ such that e and m are F-coalgebra homomorphisms, using that F preserves $\mathscr{M}$-morphisms and the unique diagonal fill-in property (cf. Remark 2.4(2)).

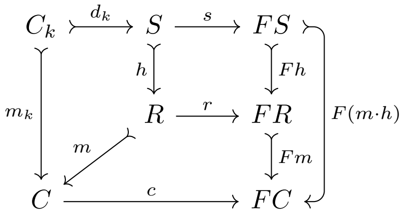
(2) For reachability, let $h\colon (S, s, i_S) \rightarrowtail (R, r, i_R)$ with $h \in \mathscr{M}$ be a pointed subcoalgebra. In the following we will define morphisms $d_k\colon C_k \rightarrowtail S$ (in $\mathscr{M}$) satisfying

$$\begin{array}{ccc} C_k & \xrightarrow{m_k} & C \\ {\scriptstyle d_k}\downarrow & & \uparrow{\scriptstyle m} \\ S & \xrightarrow{h} & R \end{array} \qquad \text{for all } k \in \mathbb{N}. \tag{5.5}$$

We define $d_0\colon C_0 \rightarrowtail S$ using the diagonal fill-in property; in fact, in the diagram below the outside commutes since $m_0 \cdot i_{C'} = i_C$ is the pointing of C and $m \cdot h$ preserves pointings:

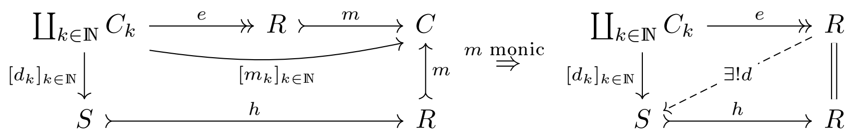


Given $d_k\colon C_k \rightarrowtail S$, note that the following diagram commutes:



The commutativity of its outside means that $(S, s \cdot d_k, m \cdot h)$ competes with the bound $m_{k+1}\colon C_{k+1} \rightarrowtail C$ of $c{\cdot}m_k$. Thus, there exists a morphism $d_{k+1}\colon C_{k+1} \rightarrowtail S$ with $m \cdot h \cdot d_{k+1} = m_{k+1}$.

Putting all squares (5.5) together, we see that the diagram on the left below commutes:



Since $m \in \mathscr{M}$ is monomorphic the outside of the diagram on the right above commutes, and we apply the diagonal fill-in property again to see that h is a split epimorphism. Since we also know that $h \in \mathscr{M}$ is a monomorphism, it is an isomorphism, whence (R, r, i_R) is reachable as desired. □

Definition 5.21. We call the above I-pointed coalgebra (R, r, i_R) the *reachable part* of (C, c, i_C).

Remark 5.22. Note that it follows from an easy lattice theoretic argument that for every join-preserving map $\varphi\colon L \to L$ on a complete lattice L, and every $\ell \in L$ the least fixed point of $\ell \vee \varphi(-)$ is given by the join

$$\bigvee_{i \in \mathbb{N}} \varphi^i(\ell). \tag{5.6}$$

Indeed, to see this recall that, by Kleene's fixed point theorem, the least fixed point of $\ell \vee \varphi(-)$ is the join of the ω-chain given by $x_0 = \bot$, the least element of L, and $x_{n+1} = \ell \vee \varphi(x_n)$. We will show by induction that

$$x_n = \bigvee_{i<n} \varphi^i(\ell) \qquad \text{for all } n < \omega.$$

The base case $n = 0$ is clear since the empty join is $\bot$, and for the induction step we use that φ preserves joins to compute:

$$x_{n+1} = \ell \vee \varphi(x_n) = \ell \vee \varphi\big(\bigvee_{i<n} \varphi^i(\ell)\big) = \ell \vee \bigvee_{i<n} \varphi^{i+1}(\ell) = \bigvee_{i<n+1} \varphi^i(\ell).$$

Applying this to $\varphi = \ominus$ we see that our Construction 5.19 of the reachable part coincides with the least fixed point of $i_0 \vee \ominus(-)$ considered by Barlocco et al. (cf. Remark 5.17).

If we additionally assume that F preserves inverse images, then the reachability construction enjoys further strong properties:

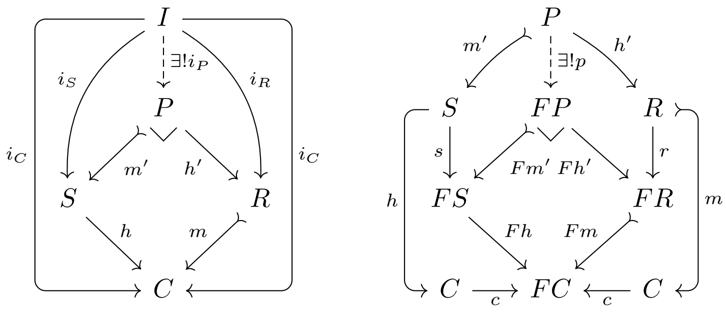
Theorem 5.23. *Suppose that F preserves inverse images. Then the full subcategory of $\mathsf{Coalg}_I(F)$ given by all reachable coalgebras is coreflective.*

Proof. Let (C, c, i_C) be an I-pointed F-coalgebra and let, $m\colon (R, r, i_R) \rightarrowtail (C, c, i_C)$ be its reachable part. We will show that this is a coreflection by verifying the corresponding universal property.

Suppose we have a homomorphism $h\colon (S, s, i_S) \to (C, c, i_C)$ where (S, s, i_S) is reachable. We need to show that h factorizes uniquely through m. Uniqueness is clear since m is monic. For existence, we form the inverse image of m under h, i.e. we form the following pullback:

$$\begin{array}{ccc} P & \xrightarrow{h'} & R \\ {\scriptstyle m'}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ S & \xrightarrow[h]{} & C \end{array} \tag{5.7}$$

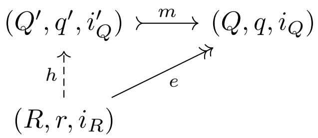
We equip P with an I-pointing and a coalgebra structure making m' and h' homomorphisms of I-pointed coalgebras. Indeed, since $m \cdot i_R = i_C = h \cdot i_S$, we obtain a pointing i_P on P, and since F preserves inverse images we also obtain a coalgebra structure p:



Clearly, this definition of (P, p, i_P) makes m' and h' homomorphisms. Since (S, s, i_S) is reachable and m' is monomorphic, the latter must be an isomorphism. Thus, $h' \cdot (m')^{-1}$ is the desired factorization of h through m, cf. (5.7). □

Corollary 5.24. *If F preserves inverse images, then reachable F-coalgebras are closed under quotients.*

Proof. Given a reachable coalgebra (R, r, i_R) and $e\colon (R, r, i_R) \twoheadrightarrow (Q, q, i_Q)$ in $\mathsf{Coalg}_I(F)$ carried by an $\mathscr{E}$-morphism. Suppose that $m\colon (Q', q', i'_Q) \to (Q, q, i_Q)$, $m \in \mathscr{M}$, is the inclusion of the reachable part of Q. By Theorem 5.23, e factorizes through m:



Using the diagonal fill-in property, we obtain a unique homomorphism $d\colon (Q, q, i_Q) \to (Q', q', i'_Q)$ such that $h = d \cdot e$ and $m \cdot d = \mathsf{id}$. This implies that m is a split epimorphism, and since $m \in \mathscr{M}$ is a monomorphism, it is an isomorphism. □

Example 5.25. For functors not preserving inverse images, reachable coalgebras need not be closed under quotients. For example, recall the functor $R\colon \mathsf{Set} \to \mathsf{Set}$ from Example 4.5 and consider the coalgebras $c\colon C \to RC$ with $C = \{x, y, z\}$ and $c(x) = (y, z)$ and $c(y) = c(z) = *$ and $d\colon D \to RD$ with $D = \{x', y'\}$ and $d(x') = d(y') = *$. Then (D, d) is a quotient of (C, c) via the coalgebra homomorphism q with $q(x) = x'$ and $q(y) = q(z) = y'$. However, (C, c, x) is reachable whereas (D, d, x') is not.

Note that, in the light of the proof of Corollary 5.24, this example also shows that reachable F-coalgebras need not form a coreflective subcategory if F does not preserves inverse images.

Finally, let us come back to $\mathscr{C} = \mathsf{Set}$ and canonical graphs. We call the subset $m_k : C_k \hookrightarrow C$ of Construction 5.19 the k^{th} *step* of the construction of the reachable part.

Corollary 5.26. *Let $F : \mathsf{Set} \to \mathsf{Set}$ preserve intersections. Then the k^{th} steps of the constructions of the reachable parts of an I-pointed coalgebra and its canonical graph are the same.*

Indeed, this follows by an easy induction from the fact that for a coalgebra $c : C \to FC$ and a subset $s : S \hookrightarrow C$, $\ominus s$ may be computed on the canonical graph of (C, c) (see Proposition 5.16).

Remark 5.27. (1) From Corollary 5.26 we can conclude that for an intersection preserving set functor F, the reachable part of a given I-pointed F-coalgebra (C, c, i_C) may be computed by a standard graph algorithm such as breadth-first-search. We thus obtain an efficient and generic algorithm for reachability of coalgebras for intersection preserving set functors.

(2) Moreover, the k^{th} subset $C_k \hookrightarrow C$ contains precisely all the states of C that are reachable along a path of length precisely k in the canonical graph of C from a state in the image C_0 of the I-pointing $i_C : I \to C$.

6 Reachability in a Kleisli Category

In this section, we present a reduction from the reachability construction in a Kleisli category for a monad on $\mathscr{C}$ to the reachability construction in the base category $\mathscr{C}$. This makes our construction applicable in Kleisli categories that fail to have an $(\mathscr{E}, \mathscr{M})$-factorization system for the desired class $\mathscr{M}$ of monomorphisms that determines the notion of subcoalgebra. Coalgebras over Kleisli categories are used to study the trace semantics of various kinds of state-based systems, see e.g. Hasuo, Jacobs, and Sokolova [14].

For our reduction, we need that finite coproducts in $\mathscr{C}$ are well behaved. In fact, recall [11] that a category is called *extensive* if it has finite coproducts and for every pair A, B of objects the canonical functor $\mathscr{C}/A \times \mathscr{C}/B \to \mathscr{C}/(A + B)$ is an equivalence of categories.

Remark 6.1. We further recall a few properties of extensive categories from [11].

(1) First note that a category with finite coproducts is extensive if and only if it has pullbacks along coproduct injections and for every commutative diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\ {\scriptstyle h_1}\big\downarrow & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle h_2} \\ B & \xrightarrow{\mathsf{inl}} & B + C & \xleftarrow{\mathsf{inr}} & C \end{array}$$

we have that $A_1 \xrightarrow{a_1} A \xleftarrow{a_2} A_2$ is a coproduct iff both squares are pullbacks.

(2) In an extensive category $\mathscr{C}$, the coproduct injections are monomorphisms, and coproducts are *disjoint*, i.e. the intersection of the subobjects $\mathsf{inl}\colon A \rightarrowtail A+B$ and $\mathsf{inr}\colon B \rightarrowtail A+B$ is $0 \rightarrowtail A+B$, where 0 is the initial object.

Example 6.2. Many categories with set-like coproducts are extensive, for example the category Set itself, as well as the categories of partially ordered sets, nominal sets, and graphs as well as every category of presheaves. In addition, the categories of unary algebras and of Jónsson-Tarski algebras (i.e. algebras A with one binary operation $A \times A \to A$ that is an isomorphism) are extensive. More generally, every topos is extensive.

In contrast, the category of monoids is not extensive.

Recall out terminology and notation for morphisms in a Kleisli category for a monad from Remark 5.3. Furthermore, recall that a monad (T, μ, η) is called *consistent* if η_X is a monomorphism for every object X of $\mathscr{C}$ [1, Definition IV.2]. Note that on Set all but two monads are consistent. In fact, only the monad C_1 mapping all sets to 1 and C_{01} mapping non-empty sets to 1 and the empty set to itself are inconsistent (see e.g. [1, Lemma IV.3]).

The following terminology is borrowed from functional programming:

Definition 6.3. A Kleisli morphism $f\colon X \multimap\!\!\to Y$ is called *pure* if $f\colon X \to TY$ factorizes through $\eta_Y\colon Y \to TY$.

A coalgebra homomorphism between coalgebras for a functor on $\mathcal{K}\ell(T)$ which is pure is called *pure coalgebra homomorphism*.

Remark 6.4. The pure morphisms form a subcategory of $\mathcal{K}\ell(T)$, and if T is consistent, then this subcategory can be identified with the base category $\mathscr{C}$ via the canonical functor $J\colon \mathscr{C} \hookrightarrow \mathcal{K}\ell(T)$ given by

$$J(f\colon X \to Y) = (\eta_Z \cdot f)\colon X \multimap\!\!\to Y.$$

Consequently, we write $g\colon X \to Y$ for pure morphisms in diagrams in $\mathcal{K}\ell(T)$ and omit the explicit application of the functor J.

Recall that an *extension* of a functor $F\colon \mathscr{C} \to \mathscr{C}$ to $\mathcal{K}\ell(T)$ is a functor $\bar{F}\colon \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$ such that the square below commutes:

$$\begin{array}{ccc} \mathcal{K}\ell(T) & \xrightarrow{\bar{F}} & \mathcal{K}\ell(T) \\ {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle J} \\ \mathscr{C} & \xrightarrow{F} & \mathscr{C} \end{array}$$

If T is consistent, then a functor on $\mathcal{K}\ell(T)$ is an extension iff it preserves pure morphisms.

Example 6.5. (1) We have already mentioned that the category Rel is the Kleisli category of the power-set monad $\mathcal{P}$. Coalgebras over $\mathcal{K}\ell(\mathcal{P})$ are systems with non-deterministic branching. For example, non-determistic automata with input alphabet Σ are coalgebras for $FX = 1 + \Sigma \times X$ (see Example 2.2(2)).

(2) For the finite power-set monad $\mathcal{P}_f X = \{S \subseteq X \mid S \text{ finite}\}$, coalgebras on $\mathcal{K}\ell(\mathcal{P}_f)$ are finitely branching non-deterministic systems. For example, finitely branching transition systems with label alphabet Σ are coalgebras for the functor $FX = \Sigma \times X$ on $\mathcal{K}\ell(\mathcal{P}_f)$.

(3) Consider the Kleisli category of the monad $\mathbb{S}^{(-)}$ given by a semiring $\mathbb{S}$ (see Example 5.4(4)). The 1-pointed coalgebras for the functor $\bar{F}X = 1 + \Sigma \times X$ are weighted automata with the input alphabet Σ. Indeed, a coalgebra structure

$$c\colon C \to \mathbb{S}^{(1+\Sigma\times C)} \cong \mathbb{S} \times \mathbb{S}^{(\Sigma\times C)}$$

assigns to each state x an output value in $\mathbb{S}$, and for every $x \in X$ it yields a map $t_x\colon \Sigma \times C \to \mathbb{S}$ where $t_x(a, y) = s$ means that there is a transition from x to y with label $a \in \Sigma$ and weight $s \in \mathbb{S}$.

(4) For the distribution monad $\mathcal{D}$, coalgebras on $\mathcal{K}\ell(\mathcal{D})$ have probabilistic branching. For example, for the functor $\bar{F}X = \Sigma\times X$ are labelled Markov chains. Indeed, a coalgebra $c\colon C \to \mathcal{D}(\Sigma\times C)$ assigns to a state x a distribution over pairs of labels and next states.

Assumption 6.6. We assume that $\mathscr{C}$ is an extensive category, that (T, μ, η) is a consistent monad on $\mathscr{C}$, and that $\bar{F}\colon \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$ is an extension of $F\colon \mathscr{C} \to \mathscr{C}$. Furthermore we consider the class $\mathscr{M}$ of all *pure monomorphisms*:

$$\mathscr{M} := \{m\colon X \to Y \mid m \text{ is a monomorphism in } \mathscr{C}\}.$$

Remark 6.7. The notions of subobjects and of subcoalgebras in $\mathcal{K}\ell(T)$ are understood w.r.t. the class $\mathscr{M}$, i.e. a subobject is represented by a Kleisli morphism $m\colon S \mathrel{-\!\!\circ\!\!\to} X$ in $\mathscr{M}$ and a subcoalgebra by a coalgebra homomorphism $h\colon (S, s) \to (C, c)$ carried by a Kleisli morphism in $\mathscr{M}$.

Notation 6.8. We write $\mathsf{Coalg}_I^{\mathsf{p}}(\bar{F})$ for the subcategory of $\mathsf{Coalg}_I(\bar{F})$ given by pure coalgebra homomorphisms.

Construction 6.9. For every I-pointed coalgebra $I \xrightarrow{i_C} C \xrightarrow{c} \bar{F}C$ in $\mathcal{K}\ell(T)$ we form the following $TF + T$-coalgebra in $\mathscr{C}$:

$$C + I \xrightarrow{c+i_C} TFC + TC \xrightarrow{TF\,\mathsf{inl} + T\,\mathsf{inl}} (TF + T)(C + I)$$

together with the I-pointing $\mathsf{inr}\colon I \to C + I$. This defines the object assignment of a functor $G\colon \mathsf{Coalg}_I^{\mathsf{p}}(\bar{F}) \to \mathsf{Coalg}_I(TF + T)$ that maps a pure coalgebra homomorphism $h\colon (C, c, i_C) \to (D, d, i_D)$ to $Gh = h + \mathsf{id}_I$.[3]

In fact, Gh is a homomorphism of I-pointed coalgebras for $TF + T$ on $\mathscr{C}$ as shown by the following commutative diagram:

$$\begin{array}{ccccccc}
I & \xrightarrow{\mathsf{inr}} & C + I & \xrightarrow{c+i_C} & TFC + TC & \xrightarrow{TF\,\mathsf{inl} + T\,\mathsf{inl}} & TF(C + I) + T(C + I) \\
 & \searrow^{\mathsf{inr}} & \downarrow{\scriptstyle h+I} & & \downarrow{\scriptstyle TFh+Th} & & \downarrow{\scriptstyle (TF+T)(h+I)} \\
 & & D + I & \xrightarrow{d+i_D} & TFD + TD & \xrightarrow{TF\,\mathsf{inl} + T\,\mathsf{inl}} & TF(D + I) + T(D + I)
\end{array}$$

[3]The fact that η is monic is used here to ensure that we may regard h as a morphism of $\mathscr{C}$.

Proposition 6.10. *The functor G reflects isomorphisms and it preserves and reflects subcoalgebras.*

Proof. Let $h\colon (C, c, i_C) \to (D, d, i_D)$ be a morphism in $\mathsf{Coalg}^{\mathsf{p}}_I(\bar{F})$. If h is in $\mathscr{M}$, then it is a monomorphism in $\mathscr{C}$ and, moreover, so is $Gh = h + \mathsf{id}_I$ since monomorphisms are closed under coproducts in the extensive category $\mathscr{C}$. This shows that G preserves subcoalgebras.

To see that it reflects them assume that $h + \mathsf{id}_I$ is a monomorphism in $\mathscr{C}$. Then we have that $\mathsf{inl} \cdot h = (h + \mathsf{id}_I) \cdot \mathsf{inl}$ is monomorphic since inl is so. Thus h is a monomorphism in $\mathscr{C}$, whence it is a pure monomorphism in $\mathcal{K}\ell(T)$.

We proceed to proving that G reflects isomorphisms. Consider $h\colon (C, c, i_C) \to (D, d, i_D)$ in $\mathsf{Coalg}^{\mathsf{p}}_I(\bar{F})$ such that $Gh = h + \mathsf{id}_I$ is an isomorphism is $\mathscr{C}$. By extensivity, we have the pullback

$$
\begin{array}{ccc}
C & \xrightarrow{\mathsf{inl}} & C + I \\
{\scriptstyle h}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle h+I} \\
D & \xrightarrow{\mathsf{inl}} & D + I
\end{array}
$$

Thus, h is an isomorphism in $\mathscr{C}$, whence in $\mathcal{K}\ell(T)$. $\square$

Lemma 6.11. *Suppose that T and F preserve finite intersections, and let $h\colon (D, d, i_D) \to G(C, c, i_C)$ be a morphism in $\mathsf{Coalg}_I(TF + T)$ carried by a monomorphism in $\mathscr{C}$. Then there exists a morphism $g\colon (E, e, i_E) \to (C, c, i_C)$ in $\mathsf{Coalg}^{\mathsf{p}}_I(\bar{F})$ such that $(D, d, i_D) = G(E, e, i_E)$ and $h = Gg$.*

Proof. Consider a morphism h in $\mathsf{Coalg}_I(TF + T)$ which is monomorphic in $\mathscr{C}$, i.e. the following diagram commutes

$$
\begin{array}{ccccc}
I \overset{i_D}{\rightarrowtail} D & & \xrightarrow{\quad d \quad} & & TFD + TD \\
\quad {\scriptstyle \mathsf{inr}}\searrow \quad \big\downarrow{\scriptstyle h} & & & & \big\downarrow{\scriptstyle TFh+Th} \\
C + I & \xrightarrow{c+i_C} & TFC + TC & \xrightarrow{TF\,\mathsf{inl} + T\,\mathsf{inl}} & TF(C+I) + T(C+I)
\end{array}
\tag{6.1}
$$

Form the pullbacks of the coproducts injections of $C + I$ along h:

$$
\begin{array}{ccccc}
E & \overset{m}{\rightarrowtail} & D & \overset{i_D}{\leftarrowtail} & I \\
{\scriptstyle g}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle h} & \llcorner & \big\| \\
C & \overset{\mathsf{inl}}{\rightarrowtail} & C + I & \overset{\mathsf{inr}}{\leftarrowtail} & I
\end{array}
\tag{6.2}
$$

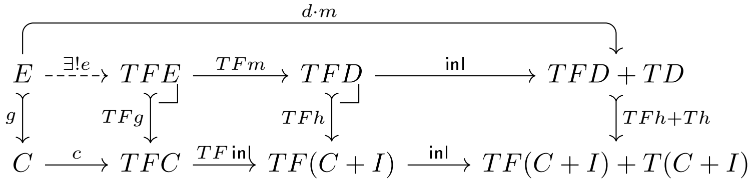
In order to see that the right-hand square is indeed a pullback, suppose we are given morphisms $p\colon X \to D$ and $q\colon X \to I$ such that $h \cdot p = \mathsf{inr} \cdot q$. Then we have

$$h \cdot i_D \cdot q = \mathsf{inr} \cdot q = h \cdot p,$$

and therefore we have $i_D \cdot q = p$ since h is monomorphic. It follows that q is the unique mediating morphism.

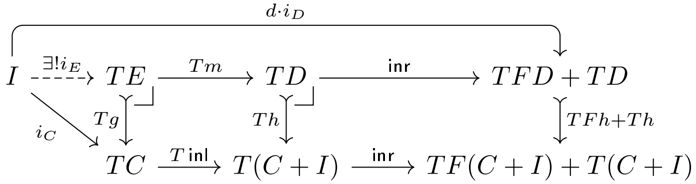
Thus, by extensivity, we have $D = E + I$ with coproduct injections m and i_D and hence, $h = g + \mathsf{id}_I$.

Note that the two pullbacks in (6.2) are in fact intersections. Since T and F preserve finite intersections, the middle square in the next diagram is a pullback, and so is the right-hand one, by extensivity. By combining (6.1) and (6.2) we see that the outside of the following diagram commutes, and so we obtain the morphism $e\colon E \to TFE$ as indicated:



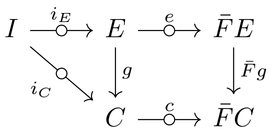
(6.3)

Similarly, we obtain i_E:

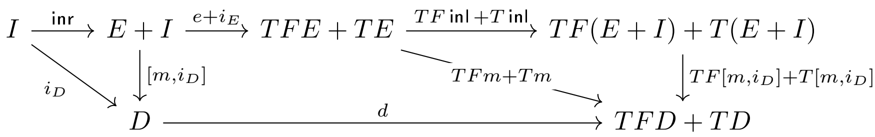


(6.4)

Thus, we have seen that $g\colon (E, e, i_E) \to (C, c, i_C)$ is a morphism in $\mathsf{Coalg}^{\mathsf{p}}_I(\bar{F})$, i.e. the diagram below commutes in $\mathcal{K}\ell(T)$:



Finally, we establish that $G(E, e, i_E) = (D, d, i_D)$ by showing that the isomorphism $[m, i_D]$ (in $\mathscr{C}$) is a homomorphism from $G(E, e, i_E)$ to (D, d, i_D) in $\mathsf{Coalg}_I(TF + T)$:



Indeed, the two triangles trivially commute, and for the middle part consider the coproduct components separately: in fact, the left- and right-hand components are the upper parts of (6.3) and (6.4), respectively. This completes the proof. □

Corollary 6.12. *Let T and F preserve finite intersections.*

(1) *The functor G preserves and reflects reachable coalgebras. That is, $G(C, c, i_C)$ is reachable iff so is (C, c, i_C).*

(2) *The reachable part of an I-pointed $\bar{F}$-coalgebra (C, c, i_C) is (up to isomorphism) given by the reachable part of $G(C, c, i_C)$.*

It follows that in order to construct the reachable part of an I-pointed $\bar{F}$-coalgebra (C, c, i_C) one may proceed as follows:

(1) Construct the reachable part of $G(C, c, i_C)$ in $\mathscr{C}$, and call the carrier D.

(2) Then $D = E + I$ for some subobject $m\colon E \rightarrowtail C$.

(3) E carries an I-pointing $i_E\colon I \multimap E$ and a $\bar{F}$-coalgebra structure $e\colon E \multimap \bar{F}E$ such that $m\colon (E, e, i_E) \rightarrowtail (C, c, i_C)$ is the reachable part.

Note that if $\mathcal{K}\ell(T)$ and $\bar{F}$ fulfill Assumption 5.10, then this gives the same result as performing Construction 5.19 directly on (C, c, i_C) in $\mathcal{K}\ell(T)$ because the reachable part of a coalgebra is unique up to isomorphism.

Proof. (1) For reflection, let $G(C, c, i_C)$ be reachable. By Proposition 6.10, every subcoalgebra $m\colon (S, s, i_S) \rightarrowtail (C, c, i_C)$ is preserved by G, thus Gm is an isomorphism, whence m is one.

For preservation, consider a subcoalgebra $m\colon (D, d, i_D) \rightarrowtail G(C, c, i_C)$. By Lemma 6.11, there exists $m'\colon (E, e, i_E) \to (C, c, i_C)$ in $\mathsf{Coalg}_I^{\mathsf{p}}(\bar{F})$ such that $m = Gm'$. Since G reflects subcoalgebras by Proposition 6.10, m' is a subcoalgebra. Finally, since (C, c, i_C) is reachable, m' is an isomorphism, thus so is m.

(2) This follows from point (1) noting that (C, c, i_C) has a unique reachable subcoalgebra since G reflects isomorphisms by Proposition 6.10. □

Remark 6.13. Observe that in the case where the base category is Set one may drop the assumption that T and F preserves finite intersections. Indeed, if I is the empty set, then the reachable part of every coalgebra is the empty subcoalgebra (in both $\mathscr{C}$ and $\mathcal{K}\ell(T)$), and so the statement is trivial (cf. Remark 2.6). And if I is non-empty, then the intersections computed in the above proof are non-empty and thus preserved by every T and F, see Trnková [25].

7 Conclusions and Future Work

We have presented a new iterative construction of the reachable part of a given I-pointed coalgebra. Our construction works for coalgebras for intersection-preserving endofunctors over a category $\mathscr{C}$ which has coproducts and a factorization system $(\mathscr{E}, \mathscr{M})$ where $\mathscr{M}$ consists of monomorphisms. For coalgebras over Set we saw that their reachable part can be constructed by running the standard breadth-first search algorithm on their canonical graph. Finally, we have considered coalgebras over Kleisli categories for a consistent finite-intersection preserving monad T. We have shown that for a functor $\bar{F}$ on $\mathcal{K}\ell(T)$ extending a finite-intersection preserving functor F on $\mathscr{C}$, the reachable part of a given I-pointed coalgebra can be obtained from the reachable part of an I-pointed $(TF + T)$-coalgebra canonically constructed from the given one.

There remain a number of questions for future work. First, it should be interesting to see whether our results still hold if we drop our assumption that $\mathscr{M}$ is a class of monomorphisms. Secondly, it seems that the reachability construction can be further generalized from working with (operators on) subcoalgebras to working with fibrations, with the subobject fibration yielding the present level of generality. Finally, we have seen that breadth-first search is an instance of our reachability construction. A fibrational approach might provide other breadth-first search based algorithms such as Dijkstra's algorithm for shortest paths and Prim's algorithm for minimum spanning trees as special instances.

References

[1] Jiří Adámek, Nathan Bowler, Paul B. Levy, and Stefan Milius. Coproducts of monads on Set. In *Proceedings of the 27th Annual Symposium on Logic in Computer Science (LICS'12)*, 2012.

[2] Jiří Adámek, Horst Herrlich, and George E. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. Dover Publications, 2nd edition, 2009.

[3] Jiří Adámek, Stefan Milius, and Lawrence S. Moss. Fixed points of functors. *Journal of Logical and Algebraic Methods in Programming*, 95:41–81, 2018.

[4] Jiří Adámek, Stefan Milius, Lawrence S. Moss, and Lurdes Sousa. Well-pointed coalgebras. *Logical Methods in Computer Science*, 9(3:2):51 pp., 2013.

[5] Jiří Adámek, Stefan Milius, Lurdes Sousa, and Thorsten Wißmann. On finitary functors and finitely presentable algebras. Draft on `https://arxiv.org/abs/1902.05788`, 2019.

[6] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1994.

[7] Jiří Adámek and Věra Trnková. *Automata and Algebras in Categories*, volume 37 of *Mathematics and its Applications*. Kluwer Academic Publishers, 1990.

[8] Simone Barlocco, Clemens Kupke, and Jurriaan Rot. Coalgebra learning via duality. In *Proceedings of the 22nd International Conference on Foundations of Software Science and Computation Structures (FoSSaCS'19)*, To appear.

[9] Michael Barr. Terminal coalgebras in well-founded set theory. *Theoretical Computer Science*, 114(2):299–315, 1993.

[10] Alwin Block. Interaction, observation and denotation. Master's thesis, Universiteit van Amsterdam, 2012.

[11] Aurelio Carboni, Steve Lack, and Robert F. C. Walters. Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, 84(2):145–158, 1993.

[12] Murdoch Gabbay and Andrew M. Pitts. A new approach to abstract syntax involving binders. In *Proceedings of the 14th Annual Symposium on Logic in Computer Science (LICS'99)*, 1999.

[13] H. Peter Gumm. From T-coalgebras to filter structures and transition systems. In *Proceedings of the 1st Conference on Algebra and Coalgebra in Computer Science (CALCO'05)*, 2005.

[14] Ichiro Hasuo, Bart Jacobs, and Ana Sokolova. Generic trace semantics via coinduction. *Logical Methods in Computer Science*, 3(4:11):1–36, 2007.

[15] Bart Jacobs. The temporal logic of coalgebras via galois algebras. *Mathematical Structures in Computer Science*, 12(6):875–903, 2002.

[16] André Joyal, Mogens Nielsen, and Glynn Winskel. Bisimulation from Open Maps. *Information and Computation*, 127(2):164–185, 1996.

[17] Alexander Kurz, Daniela Petrisan, Paula Severi, and Fer-Jan de Vries. Nominal coalgebraic data types with applications to lambda calculus. *Logical Methods in Computer Science*, 9(4), 2013.

[18] Sławomir Lasota. Coalgebra morphisms subsume open maps. *Theoretical Computer Science*, 280(1-2):123–135, 2002.

[19] Zohar Manna and Amir Pnüeli. *The Temporal Logic of Reactive and Concurrent Systems: Specification*. Springer-Verlag, 1992.

[20] Stefan Milius, Lutz Schröder, and Thorsten Wißmann. Regular behaviours with names: On rational fixpoints of endofunctors on nominal sets. *Applied Categorical Structures*, 24(5):663–701, 2016.

[21] Stefan Milius and Thorsten Wißmann. Finitary corecursion for the infinitary lambda calculus. In *Proceedings of the 6th Conference on Algebra and Coalgebra in Computer Science (CALCO'15)*, 2015.

[22] Andrew M. Pitts. *Nominal Sets: Names and Symmetry in Computer Science*, volume 57 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2013.

[23] Jan J. M. M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249(1):3–80, 2000.

[24] Lutz Schröder, Dexter Kozen, Stefan Milius, and Thorsten Wißmann. Nominal automata with name binding. In *Proceedings of the 20th International Conference on Foundations of Software Science and Computation Structures (FoSSaCS'17)*, 2017.

[25] Věra Trnková. Some properties of set functors. *Commentationes Mathematicae Universitatis Carolinae*, 10(2):323–352, 1969.

[26] Věra Trnková. On a descriptive classification of set functors I. *Commentationes Mathematicae Universitatis Carolinae*, 12(1):143–174, 1971.

[27] Thorsten Wißmann, Jérémy Dubut, Shin-ya Katsumata, and Ichiro Hasuo. Path category for free - open morphisms from coalgebras with non-deterministic branching. In *Proceedings of the 22nd International Conference on Foundations of Software Science and Computation Structures (FoSSaCS'19)*, To appear, full version on `http://arxiv.org/abs/1811.12294`.